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Starlikeness of meromorphic functions involving certain differential inequalities

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Abstract: In the present paper, we define a class of non-Bazilevic functions in punctured unit disk and study a differential inequality to obtain certain new criteria for starlikeness of meromorphic functions.

Keywords: Meromorphic function; Meromorphic starlike function.

MSC: 30C45; 30C80.

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1. Introduction

Let Σ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{0}^{\infty} a_n z^n,$$

which are analytic in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z : |z| < 1\}$. A function $f \in \Sigma$ is said to be meromorphic starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$ and the class of meromorphic starlike functions is denoted by $\mathcal{MS}^* = \mathcal{MS}^*(0)$.

In the theory of meromorphic functions, many authors have obtained different sufficient conditions for meromorphically starlike functions. Some of them are stated below:

Kargar *et al.*, [1] proved the following results:

Theorem 1. *Assume that $f(z) \neq 0$ for $\mathbb{E}_0$. If $f \in \Sigma(p)$ satisfies*

$$\left|\frac{1}{\sqrt[p]{f(z)}}\left(\frac{f'(z)}{f(z)} +\right) + p\right| < p\lambda(\beta)|b(z)|,\ z \in \mathbb{E}_0,$$

then f is a p-valently meromorphic strongly-starlike of order β.

Theorem 2. *Assume that $f(z) \neq 0$ for $\mathbb{E}_0$. If $f \in \Sigma$ satisfies*

$$\left|\left(\frac{f(z)}{z^{-\alpha}}\right)^{\frac{1}{\alpha-1}}\left(\frac{f'(z)}{f(z)} + \frac{\alpha}{z}\right) + 1 - \alpha\right| < \frac{2}{\sqrt{5}},\ z \in \mathbb{E}_0,$$

then f is meromorphic starlike function of order α.

Goswami *et al.*, [2] proved the following results:

Theorem 3. *If $f(z) \in \Sigma_p, n$ with $f(z) \neq 0$ for all $z \in \mathbb{E}_0$, satisfies the following inequality*

$$\left|[z^p f(z)]^{\frac{1}{\alpha-p}} \left(\frac{zf'(z)}{f(z)} + \alpha\right) + p - \alpha\right| < \frac{(n+1)(p-\alpha)}{\sqrt{(n+1)^2+1}}, z \in \mathbb{E},$$

for some real values of α $(0 \leq \alpha < p)$, then $f \in \mathcal{MS}^*_{p,n}(\alpha)$.

Theorem 4. *If $f(z) \in \Sigma_p, n$ with $f(z) \neq 0$ for all $z \in \mathbb{E}_0$ satisfies the following inequality*

$$\left|\frac{\gamma[z^p f(z)]^\gamma}{z} \left(\frac{zf'(z)}{f(z)} + p\right)\right| \leq \frac{(n+1)}{2\sqrt{(n+1)^2+1}}, z \in \mathbb{E},$$

for $\gamma \leq -\frac{1}{p}$, then $f \in \mathcal{MS}^*_{p,n}\left(p + \frac{1}{\gamma}\right)$.

In [3], Sahoo *et al.*, investigated a new class $\mathcal{U}_n(\alpha, \lambda, \mu)$, of non-Bazilevic analytic functions by

$$\mathcal{U}_n(\alpha, \lambda, \mu) = \left\{ f \in \mathcal{A}_n : \left|(1-\alpha)\left(\frac{z}{f(z)}\right)^\mu + \alpha f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda,\ z \in \mathbb{E} \right\}.$$

For different choices of μ with $\alpha = 1$, many authors has studied this class which are included in [4–6]. In this paper, we define above class of non-Bazilevic functions in punctured unit disk and study a differential inequality to obtain certain new criteria for starlikeness of meromorphic functions.

2. Main results

To prove our main result, we shall make use of following lemma of Hallenback and Ruscheweyh [7].

Lemma 1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, is analytic in $\mathbb{E}$ nd $F \prec G$, then*

$$\frac{1}{z^\gamma} \int_0^z F(w) w^{\gamma-1} dw \prec \frac{1}{nz^{\frac{\gamma}{n}}} \int_0^z G(w) w^{\frac{\gamma}{n}-1} dw.$$

Theorem 5. *Let α, β, δ be real numbers such that $\alpha < \frac{2}{\delta - 1}$, $\beta > 0$, $0 \leq \delta < 1$ and let*

$$0 < M \equiv M(\alpha, \beta, \delta) = \frac{(\beta - \alpha)[2 + \alpha(1-\delta)]}{\alpha[1 + \beta(1-\delta)]}. \tag{1}$$

If $f \in \Sigma$ satisfies the differential inequality

$$\left|\left(\frac{1}{zf(z)}\right)^\beta \left(1 + \alpha + \alpha\frac{zf'(z)}{f(z)}\right) - 1\right| < M(\alpha, \beta, \delta),\ z \in \mathbb{E}, \tag{2}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Proof. Let us define

$$\left(\frac{1}{zf(z)}\right)^\beta = u(z),\ z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain

$$\frac{zf'(z)}{f(z)} = -\left(1 + \frac{zu'(z)}{\beta u(z)}\right). \tag{3}$$

Therefore, in view of (3), we have

$$u(z) - \frac{\alpha}{\beta} zu'(z) \prec 1 + Mz. \tag{4}$$

The use of Lemma 1 $\left(taking\ \gamma = -\frac{\beta}{\alpha}\right)$ in (4) gives

$$u(z) \prec 1 + \frac{\gamma Mz}{\gamma + 1},$$

or

$$|u(z) - 1| < \frac{\beta M}{\beta - \alpha} < 1,$$

therefore, we obtain

$$|u(z)| > 1 - \frac{\beta M}{\beta - \alpha}. \tag{5}$$

Write $-\frac{zf'(z)}{f(z)} = (1-\delta)w(z) + \delta, 0 \le \delta < 1$ and therefore (2) reduces to

$$|(1+\alpha)u(z) - \alpha u(z)[(1-\delta)w(z) + \delta] - 1| < M.$$

We need to show that $\Re(w(z)) > 0,\ z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0,\ z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix, x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|(1+\alpha)u(z_0) - \alpha u(z_0)[(1-\delta)ix + \delta] - 1| \ge M. \tag{6}$$

By making use of (3), we have

$$\begin{aligned} |(1+\alpha)u(z_0) - \alpha u(z_0)[(1-\delta)ix + \delta] - 1| &\ge |[1 + \alpha(1-\delta) - \alpha(1-\delta)ix]u(z_0)| - 1 \\ &= \sqrt{[1+\alpha(1-\delta)]^2 + \alpha^2(1-\delta)^2x^2}\ |u(z_0)| - 1 \\ &\ge |1 + \alpha(1-\delta)||u(z_0)| - 1 \\ &\ge |1 + \alpha(1-\delta)| \left(1 - \frac{\beta M}{\beta - \alpha}\right) - 1 \ge M. \end{aligned} \tag{7}$$

Now (7) is true in view of (1) and therefore, (6) holds. Hence $\Re(w(z)) > 0$, i.e.,

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ 0 \le \delta < 1,\ z \in \mathbb{E}.$$

□

Remark 1. Let α, β, δ be real numbers such that $\alpha < \frac{2}{\delta - 1}, 0 \le \delta < 1, \beta > 0$ and if $f(z) \in \Sigma$ satisfies

$$\left|\left(\frac{1}{zf(z)}\right)^{\beta}\left(\frac{1}{\alpha} + 1 + \frac{zf'(z)}{f(z)}\right) - \frac{1}{\alpha}\right| < \frac{(\beta - \alpha)[2 + \alpha(1-\delta)]}{\alpha^2[1 + \beta(1-\delta)]},$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Letting $\alpha \to \infty$ in above remark, we get the following result:

Theorem 6. *Let β, δ be real numbers such that $\beta > 0, 0 \le \delta < 1$ and let $f(z) \in \Sigma$ satisfy*

$$\left| \left(\frac{1}{zf(z)}\right)^{\beta} \left(1 + \frac{zf'(z)}{f(z)}\right) \right| < \frac{1-\delta}{1+\beta(1-\delta)}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

3. Deductions

Setting $\beta = 1$ in Theorem 5, we obtain

Corollary 1. *Let α and δ be real numbers such that $\alpha < \dfrac{2}{\delta - 1}, 0 \le \delta < 1$ and suppose that $f \in \Sigma$ satisfies*

$$\left| \frac{1}{zf(z)} \left(1 + \alpha + \alpha \frac{zf'(z)}{f(z)}\right) - 1 \right| < \frac{(1-\alpha)(2+\alpha(1-\delta))}{\alpha(2-\delta)},\ z \in \mathbb{E},$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta\ , z \in \mathbb{E},$$

i.e., $f \in \mathcal{MS}^*(\delta),\ z \in \mathbb{E}$.

Writing $\delta = 0$ in above corollary, we get the following result:

Corollary 2. *Let $f \in \Sigma$ satisfy*

$$\left| \frac{1}{zf(z)} \left(1 + \alpha + \alpha \frac{zf'(z)}{f(z)}\right) - 1 \right| < \frac{(1-\alpha)(2+\alpha)}{2\alpha},\ z \in \mathbb{E},$$

then $f \in \mathcal{MS}^*,\ z \in \mathbb{E}$.

Setting $\beta = 1$ in Theorem 6, we get the following result:

Corollary 3. *Let δ be a real number such that $0 \le \delta < 1$ and let $f(z) \in \Sigma$ satisfy*

$$\left| \frac{1}{zf(z)} \left(1 + \frac{zf'(z)}{f(z)}\right) \right| < \frac{1-\delta}{2-\delta}$$

then

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Setting $\delta = 0$ in above corollary, we get the following result:

Corollary 4. *Let $f(z) \in \Sigma$ satisfy*

$$\left| \frac{1}{zf(z)} \left(1 + \frac{zf'(z)}{f(z)}\right) \right| < \frac{1}{2}$$

then $f \in \mathcal{MS}^*,\ z \in \mathbb{E}$.

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