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Article

Certain new subclasses of m-fold symmetric bi-pseudo-starlike functions using Q-derivative operator

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Abstract: In this current study, we introduced and investigated two new subclasses of the bi-univalent functions associated with q-derivative operator; both f and f^{-1} are m-fold symmetric holomorphic functions in the open unit disk. Among other results, upper bounds for the coefficients $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ are found in this study. Also certain special cases are indicated.

Keywords: m-fold symmetric bi-univalent functions, analytic functions, univalent function.

MSC: 30C45.

1. Introduction

Let $\mathcal{A}$ be the family of holomorphic functions, normalized by the conditions $f(0) = f'(0) - 1 = 0$ which is of the form

$$f(z) = z + \rho_2 z^2 + \rho_3 z^3 + \cdots \tag{1}$$

in the open unit disk $\Omega = \{z; z \in \mathbb{C} \text{ and } |z| < 1\}$. We denote by $\mathcal{G}$ the subclass of functions in $\mathcal{A}$ which are univalent in Ω (for more details see [1]).

The Keobe-One Quarter Theorem [1] state that the image of Ω under all univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Hence all univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfy $f^{-1}(f(z))$ and $f(f^{-1}(v)) = v$ ($|v| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$), where

$$g(v) = f^{-1}(v) = v - \rho_2 v^2 + (2\rho_2^2 - \rho_3)v^3 - (5\rho_2^3 - 5\rho_2\rho_3 + \rho_4)v^4 + \cdots \tag{2}$$

A function $f \in \mathcal{A}$ denoted by Σ is said to be bi-univalent in Ω if both $f^{-1}(z)$ ans $f(z)$ are univalent in Ω (see for details [2–11]).

A domain Ψ is said to be m-fold symmetric if a rotation of Ψ about the origin through an angle $2\pi/m$ carries Ψ on itself. Therefore, a function $f(z)$ holomorphic in Ω is said to be m-fold symmetric if

$$f\left(e^{\frac{2\pi i}{m}} z\right) = e^{\frac{2\pi i}{m}} f(z).$$

A function is said to be m-fold symmetric if it has the following normalized form

$$f(z) = z + \sum_{\phi=1}^{\infty} \rho_{m\phi+1} z^{m\phi+1} \qquad (z \in \Omega, \quad m \in \mathcal{N} = \{1, 2, 3, \cdots\}). \tag{3}$$

Let $\mathfrak{S}_m$ the class of m-fold symmetric univalent functions in Ω, that are normalized by (3), in which, the functions in the class $\mathfrak{S}$ are *one*-fold symmetric. Similar to the concept of m-fold symmetric univalent functions, we introduced the concept of m-fold symmetric bi-univalent functions which is denoted by Σ_m. Each of the function $f \in \Sigma$ produces m-fold symmetric bi-univalent function for each integer $m \in \mathcal{N}$.

The normalized form of $f(z)$ is given as in (3) and the series expansion for $f^{-1}(z)$, which has been investigated by Srivastava *et al.*, [12], is given below:

$$\begin{aligned} g(v) =& f^{-1}(v) \\ =& v - \rho_{m+1} v^{m+1} + \left[(m+1)\rho_{m-1}^{2} - \rho_{2m+1} \right] v^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)\rho_{m+1}^{3} - (3m+2)\rho_{m+1}\rho_{2m+1} + \rho_{3m+1} \right]. \end{aligned} \tag{4}$$

Some of the examples of m-fold symmetric bi-univalent functions are

$$\left\{ \frac{z^m}{1-z^m} \right\}^{\frac{1}{m}},$$

$$[-\log(1-z^m)]^{\frac{1}{m}},$$

and

$$\left\{ \frac{1}{2}\log\left(\frac{1+z^m}{1-z^m} \right)^{\frac{1}{m}} \right\}.$$

For more details on m-fold symmetric analytic bi-univalent functions (see [5,12–17]).

Jackson [18,19] introduced the q-derivative operator $\mathcal{D}_q$ of a function as follows;

$$\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \tag{5}$$

and $\mathcal{D}_q f(0) = f'(0)$. In case of $g(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$\mathcal{D}_q z^k = \frac{z^k - (zq)^k}{(q-1)z} = [k]_q z^{k-1}.$$

As $q \longrightarrow 1^-$ and $k \in \mathcal{N}$, we get

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^k \longrightarrow k, \tag{6}$$

where $(z \neq 0,\ q \neq 0)$. For more details on the concepts of q-derivative (see [5,20–27]).

Definition 1. [28] Let $f(z) \in \mathcal{A},\ 0 \leq \chi < 1$ and $\sigma \geq 1$ is real. Then $f(z) \in L_\sigma(\chi)$ of σ-pseodu-starlike function of order χ in Ω if and only if

$$\Re\left(\frac{z[f'(z)]^\sigma}{f(z)} \right) > \chi. \tag{7}$$

Babalola [28] verified that, all pseodu-starlike function are Bazilevic of type $\left(1 - \frac{1}{\sigma}\right)$, order $\chi^{\frac{1}{\sigma}}$ and univalent in Ω.

Lemma 1. [1] *Let the function $\omega \in \mathcal{P}$ be given by the following series $\omega(z) = 1 + \omega_1 z + \omega_2 z^2 + \cdots \quad (z \in \Omega)$. The sharp estimate given by $|\omega_n| \leq 2 \quad (n \in \mathcal{N})$ holds true.*

In [29] Girgaonkar *et al.,* introduced a new subclasses of holomorphic and bi-univalent functions as follows:

Definition 2. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_\Sigma(\chi)$ $(0 < \chi \leq 1, (z,v) \in \Omega)$ if $f \in \mathcal{E}$, $|\arg(f'(z))^\sigma| < \frac{\chi\pi}{2}$ and $|\arg(g'(v))^\sigma| < \frac{\chi\pi}{2}$, where $g(v)$ is given by (2).

Definition 3. A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}(\psi)$ $(0 \leq \psi < 1, (z,v) \in \Omega)$ if $\vartheta \in \Sigma$, $\Re[(f'(z))^{\sigma}] > \psi$ and $\Re[(g'(v))^{\sigma}] > \psi$, where $g(v)$ is given by (2).

In this current research, we introduced two new subclasses denoted by $\mathcal{M}^{q,\sigma}_{\Sigma,m}(\chi)$ and $\mathcal{M}^{q,\sigma}_{\Sigma,m}(\psi)$ of the function class Σ_m and obtain estimates coefficient $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ for functions in these two new subclasses.

2. Main 4esults

Definition 4. A function $f(z)$ given by (3) is said to be in the class $\mathcal{M}^{q,\sigma}_{\Sigma,m}(\chi)$ $(m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z,v) \in \Omega)$ if

$$f \in \Sigma \quad and \quad |\arg(\mathcal{D}_q f(z))^{\sigma}| < \frac{\chi\pi}{2}, \tag{8}$$

and

$$|\arg(\mathcal{D}_q g(v))^{\sigma}| < \frac{\chi\pi}{2}, \tag{9}$$

where $g(v)$ is given by (2).

Remark 1. We have the class $\lim_{q\longrightarrow 1^{-1}} \mathcal{M}^{\sigma}_{\Sigma,1}(\chi) = \mathcal{M}^{\sigma}_{\Sigma}(\chi)$ which was introduced and studied by Girgaonkar *et al.*, [29].

Remark 2. We have the class $\lim_{q\longrightarrow 1^{-1}} \mathcal{M}^{1}_{\Sigma,1}(\chi) = \mathcal{M}_{\Sigma}(\chi)$ which was introduced and studied by Srivastava *et al.*, [11].

Theorem 1. *Let $f(z) \in \mathcal{M}^{q,\sigma}_{\Sigma,m}(\chi)$, $(m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z,v) \in \Omega)$ be given (3). Then*

$$|\rho_{m+1}| \leq \frac{2\chi}{\sqrt{(m+1)\sigma\chi[2m+1]_q - (\chi-\sigma)\sigma[m+1]_q^2}}, \tag{10}$$

and

$$|\rho_{2m+1}| \leq \frac{2\chi}{\sigma[2m+1]_q} + \frac{2(m+1)\chi^2}{\sigma^2[m+1]_q^2}. \tag{11}$$

Proof. Using inequalities (1) and (9), we get

$$(\mathcal{D}_q f(z))^{\sigma} = [\tau(z)]^{\chi}, \tag{12}$$

and

$$(\mathcal{D}_q g(v))^{\sigma} = [\varsigma(v)]^{\chi} \tag{13}$$

respectively, where $\tau(z)$ and $\varsigma(v)$ in $\mathcal{P}$ are given by the following series

$$\tau(z) = 1 + \tau_m z^m + \tau_{2m} z^{2m} + \tau_{3m} z^{3m} + \cdots, \tag{14}$$

and

$$\varsigma(v) = 1 + \varsigma_m v^m + \varsigma_{2m} v^{2m} + \varsigma_{3m} v^{3m} + \cdots. \tag{15}$$

Clearly,

$$[\tau(z)]^{\chi} = 1 + \chi\tau_m z^m + \left(\chi\tau_{2m} + \frac{\chi(\chi-1)}{2}\tau_m^2\right) z^{2m} + \cdots,$$

and

$$[\varsigma(v)]^{\chi} = 1 + \chi\varsigma_m v^m + \left(\chi\varsigma_{2m} + \frac{\chi(\chi-1)}{2}\varsigma_m^2\right) v^{2m} + \cdots.$$

Also

$$(\mathcal{D}_q f(z))^{\sigma} = 1 + \sigma[m+1]_q\rho_{m+1} z^m + \left(\sigma[2m+1]_q\rho_{2m+1} + \frac{\sigma(\sigma-1)}{2}[m+1]_q^2\rho_{m+1}^2\right) z^{2m} + \cdots,$$

and

$$\begin{aligned}(\mathcal{D}_q g(v))^\sigma =& 1-\sigma[m+1]_q\rho_{m+1}v^m-\sigma[2m+1]_q\rho_{2m+1}v^{2m}\\ &+\left(\sigma(m+1)[2m+1]_q\rho_{m+1}^2+\frac{\sigma(\sigma-1)}{2}[m+1]_q^2\rho_{m+1}^2\right)v^{2m}+\cdots\end{aligned}$$

Comparing the coefficients in (12) and (13), we have

$$\sigma[m+1]_q\rho_{m+1}=\chi\tau_m, \tag{16}$$

$$\sigma[2m+1]_q\rho_{2m+1}+\frac{\sigma(\sigma-1)}{2}[m+1]_q^2\rho_{m+1}^2=\chi\tau_{2m}+\frac{\chi(\chi-1)}{2}\tau_m^2, \tag{17}$$

$$-\sigma[m+1]_q\rho_{m+1}=\chi\varsigma_m, \tag{18}$$

$$-\sigma[2m+1]_q\rho_{2m+1}+\left(\sigma(m+1)[2m+1]_q+\frac{\sigma(\sigma-1)}{2}[m+1]_q^2\right)\rho_{m+1}^2=\chi\varsigma_{2m}+\frac{\chi(\chi-1)}{2}\varsigma_m^2. \tag{19}$$

From (16) and (18), we obtain

$$\tau_m=-\varsigma_m, \tag{20}$$

and

$$2\sigma[m+1]_q^2\rho_{m+1}^2=\chi^2(\tau_m^2+\varsigma_m^2). \tag{21}$$

Further from (17), (19) and (21), we obtain that

$$\sigma(\sigma-1)\chi[m+1]_q^2\rho_{m+1}^2+(m+1)\sigma\chi[2m+1]_q\rho_{m+1}^2-(\chi-1)\sigma^2[m+1]_q^2\rho_{m+1}^2=\chi^2(\tau_{2m}+\varsigma_{2m}).$$

Therefore, we have

$$\rho_{m+1}^2=\frac{\chi^2(\tau_{2m}+\varsigma_{2m})}{\sigma[m+1]_q^2(\sigma-\chi)+(m+1)\sigma\chi[2m+1]_q}. \tag{22}$$

By applying Lemma 1 for the coefficients τ_{2m} and ς_{2m}, then we have

$$|\rho_{m+1}|\leq\frac{2\chi}{\sqrt{(m+1)\sigma\chi[2m+1]_q-(\chi-\sigma)\sigma[m+1]_q^2}}.$$

Also, to find the bound on $|\rho_{2m+1}|$, using the relation (19) and (17), we obtain

$$2\sigma[2m+1]_q\rho_{2m+1}-(m+1)\sigma[2m+1]_q\rho_{m+1}^2=\chi(\tau_{2m}-\varsigma_{2m})+\frac{\chi(\chi-1)}{2}(\tau_m^2-\varsigma_m^2). \tag{23}$$

It follows from (20), (21) and (23),

$$\rho_{2m+1}=\frac{(m+1)\chi^2\tau_m^2}{2\sigma^2[m+1]_q^2}+\frac{\chi(\tau_{2m}-\varsigma_{2m})}{2\sigma[2m+1]_q}. \tag{24}$$

Applying Lemma 1 for the coefficients $\tau_m,\tau_{2m},\varsigma_m,\varsigma_{2m}$, then we have

$$|\rho_{2m+1}|\leq\frac{2\chi}{\sigma[2m+1]_q}+\frac{2(m+1)\chi^2}{\sigma^2[m+1]_q^2}.$$

□

Choosing $q\longrightarrow 1^{-1}$ in Theorem 1, we get the following result:

Corollary 1. *Let $f(z)\in\mathcal{M}^\sigma_{\Sigma,m}(\chi)$, $(m\in\mathcal{N},\sigma\geq 1,0<\chi\leq 1,(z,v)\in\Omega)$ be given (3). Then*

$$|\rho_{m+1}|\leq\frac{2\chi}{\sqrt{(m+1)[\sigma\chi m+\sigma^2 m+\sigma^2]}}, \tag{25}$$

and

$$|\rho_{2m+1}| \leq \frac{2\chi}{\sigma(2m+1)} + \frac{2\chi^2}{\sigma^2(m+1)}. \tag{26}$$

Choosing $m = 1$ (0ne-fold case) in Theorem 1, we get the following result:

Corollary 2. *Let $f(z) \in \mathcal{M}_{\Sigma}^{q,\sigma}(\chi)$, $(0 < q < 1, \sigma \geq 1, 0 < \chi \leq 1, (z,v) \in \Omega)$ be given* (1). *Then*

$$|\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma\chi[3]_q - (\chi - \sigma)\sigma[2]_q^2}}, \tag{27}$$

and

$$|\rho_3| \leq \frac{2\chi}{\sigma[3]_q} + \frac{4\chi^2}{\sigma^2[2]_q^2}, \tag{28}$$

Choosing $q \longrightarrow 1^{-1}$ in Corollary 2, we get the following result:

Corollary 3. [29] *Let $f(z) \in \mathcal{M}_{\Sigma}^{\sigma}(\chi)$, $(\sigma \geq 1, 0 < \chi \leq 1, (z,v) \in \Omega)$ be given* (1). *Then*

$$|\rho_2| \leq \frac{2\chi}{\sqrt{2\sigma(2\sigma + \chi)}}, \tag{29}$$

and

$$|\rho_3| \leq \frac{\chi(2\sigma + 3\chi)}{3\sigma^2}. \tag{30}$$

Remark 3. For one-fold case, we have $\lim_{q \longrightarrow 1^{-1}} \mathcal{M}_{\Sigma,1}^{q,1}(\chi) = \mathcal{M}_{\Sigma}(\chi)$, and we can get the results of Srivastava *et al.*, [11].

Definition 5. A function $f(z)$ given by (3) is said to be in the class $\mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi)$ $(m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z,v) \in \Omega)$ if

$$f \in \Sigma \quad and \quad \Re[(\mathcal{D}_q f(z))^{\sigma}] > \psi, \tag{31}$$

and

$$\Re[(\mathcal{D}_q g(v))^{\sigma}] > \psi, \tag{32}$$

where $g(v)$ is given by (2).

Remark 4. We have the class $\lim_{q \longrightarrow 1^{-1}} \mathcal{M}_{\Sigma,1}^{\sigma}(\psi) = \mathcal{M}_{\Sigma}^{\sigma}(\chi)$ which was introduced and studied by Girgaonkar *et al.*, [29].

Remark 5. We have the class $\lim_{q \longrightarrow 1^{-1}} \mathcal{M}_{\Sigma,1}^{1}(\psi) = \mathcal{M}_{\Sigma}(\chi)$ which was introduced and studied by Srivastava *et al.*, [11].

Theorem 2. *Let $f(z) \in \mathcal{M}_{\Sigma,m}^{q,\sigma}(\psi)$, $(m \in \mathcal{N}, 0 < q < 1, \sigma \geq 1, 0 \leq \psi < 1, (z,v) \in \Omega)$ be given* (3). *Then*

$$|\rho_{m+1}| \leq \min\left\{\frac{2(1-\psi)}{\sigma[m+1]_q}, 2\sqrt{\frac{1-\psi}{\sigma(\sigma-1)[m+1]_q^2 + (m+1)\sigma[2m+1]_q}}\right\}, \tag{33}$$

and

$$|\rho_{2m+1}| \leq \frac{2(m+1)(1-\psi)}{\sigma(\sigma-1)[m+1]_q^2 + (m+1)\sigma[2m+1]_q} + \frac{2(1-\psi)}{\sigma[2m+1]_q}. \tag{34}$$

Proof. Using inequalities (31) and (32), we get

$$(\mathcal{D}_q f(z))^{\sigma} = \psi + (1-\psi)\tau(z), \tag{35}$$

and

$$(\mathcal{D}_q g(v))^\sigma = \psi + (1-\psi)\varsigma(v), \tag{36}$$

here $\tau(z)$ and $\varsigma(v)$ in $\mathcal{P}$ are given by the following series

$$\tau(z) = 1 + \tau_m z^m + \tau_{2m} z^{2m} + \tau_{3m} z^{3m} + \cdots,$$

and

$$\varsigma(v) = 1 + \varsigma_m v^m + \varsigma_{2m} v^{2m} + \varsigma_{3m} v^{3m} + \cdots.$$

Clearly,

$$\psi + (1-\psi)\tau(z) = 1 + (1-\psi)\tau_m z^m + (1-\psi)\tau_{2m} z^{2m} + \cdots,$$

and

$$\psi + (1-\psi)\varsigma(v) = 1 + (1-\psi)\varsigma_m v^m + (1-\psi)\varsigma_{2m} v^{2m} + \cdots.$$

Also

$$(\mathcal{D}_q f(z))^\sigma = 1 + \sigma[m+1]_q \rho_{m+1} z^m + \left(\sigma[2m+1]_q \rho_{2m+1} + \frac{\sigma(\sigma-1)}{2}[m+1]_q^2 \rho_{m+1}^2\right) z^{2m} + \cdots,$$

and

$$\begin{aligned}(\mathcal{D}_q g(v))^\sigma =& 1 - \sigma[m+1]_q \rho_{m+1} v^m - \sigma[2m+1]_q \rho_{2m+1} v^{2m} \\ &+ \left(\sigma(m+1)[2m+1]_q \rho_{m+1}^2 + \frac{\sigma(\sigma-1)}{2}[m+1]_q^2 \rho_{m+1}^2\right) v^{2m} + \cdots.\end{aligned}$$

Now comparing the coefficients in (35) and (36), we get

$$\sigma[m+1]_q \rho_{m+1} = (1-\psi)\tau_m, \tag{37}$$

$$\sigma[2m+1]_q \rho_{2m+1} + \frac{\sigma(\sigma-1)}{2}[m+1]_q^2 \rho_{m+1}^2 = (1-\psi)\tau_{2m}, \tag{38}$$

$$-\sigma[m+1]_q \rho_{m+1} = (1-\psi)\varsigma_m, \tag{39}$$

$$-\sigma[2m+1]_q \rho_{2m+1} + \left(\sigma(m+1)[2m+1]_q + \frac{\sigma(\sigma-1)}{2}[m+1]_q^2\right)\rho_{m+1}^2 = (1-\psi)\varsigma_{2m}. \tag{40}$$

From (37) and (39), we obtain

$$\tau_m = -\varsigma_m, \tag{41}$$

and

$$2\sigma[m+1]_q^2 \rho_{m+1}^2 = (1-\psi)^2(\tau_m^2 + \varsigma_m^2). \tag{42}$$

Also, from (38) and (40), we get

$$\sigma(\sigma-1)\chi[m+1]_q^2 \rho_{m+1}^2 + (m+1)\sigma[2m+1]_q \rho_{m+1}^2 = (1-\psi)(\tau_{2m} + \varsigma_{2m}). \tag{43}$$

Applying the Lemma 1 for the coefficients $\tau_m, \tau_{2m}, \varsigma_m, \varsigma_{2m}$, we find that

$$|\rho_{m+1}| \leq 2\sqrt{\frac{(1-\psi)}{\sigma(\sigma-1)[m+1]_q^2 + (m+1)\sigma[2m+1]_q}}.$$

Also, to find the bound on $|\rho_{2m+1}|$, using the relation (40) and (38), we obtain

$$-(m+1)\sigma[2m+1]_q \rho_{m+1}^2 + 2\sigma[2m+1]_q \rho_{2m+1} = (1-\psi)(\tau_{2m} - \varsigma_{2m}), \tag{44}$$

or equivalently

$$\rho_{2m+1} = \frac{(1-\psi)(\tau_{2m}-\varsigma_{2m})}{2\sigma[2m+1]_q} + \frac{(m+1)}{2}\rho^2_{m+1}. \tag{45}$$

By substituting the value of ρ^2_{m+1} from (42), we have

$$\rho_{2m+1} = \frac{(1-\psi)(\tau_{2m}-\varsigma_{2m})}{2\sigma[2m+1]_q} + \frac{(m+1)(1-\psi)^2(\tau^2_m+\varsigma^2_m)}{4\sigma^2[m+1]^2_q}. \tag{46}$$

Applying the Lemma 1 for the coefficients $\tau_m, \tau_{2m}, \varsigma_m, \varsigma_{2m}$, we get

$$|\rho_{2m+1}| \leq \frac{2(1-\psi)}{\sigma[2m+1]_q} + \frac{2(m+1)(1-\psi)^2}{2\sigma^2[m+1]^2_q}.$$

Also, by using (43) and (45), and applying Lemma 1 we obtain

$$|\rho_{2m+1}| \leq \frac{2(m+1)(1-\psi)}{\sigma(\sigma-1)[m+1]^2_q + (m+1)\sigma[2m+1]_q} + \frac{2(1-\psi)}{\sigma[2m+1]_q}.$$

This complete the proof. □

Choosing $q \longrightarrow 1^{-1}$ in Theorem 2, we get the following result:

Corollary 4. *Let $f(z) \in \mathcal{M}^{\sigma}_{\Sigma,m}(\psi)$, $(m \in \mathcal{N}, \sigma \geq 1, 0 \leq \psi < 1, (z,v) \in \Omega)$ be given (3). Then*

$$|\rho_{m+1}| \leq \begin{cases} 2\sqrt{\frac{(1-\psi)}{\sigma(\sigma-1)[m+1]^2+(m+1)\sigma[2m+1]}} & 0 \leq \psi \leq \frac{m}{1+2m}, \\ \frac{2(1-\psi)}{\sigma[m+1]} & \frac{m}{1+2m} \leq \psi < 1, \end{cases}$$

and

$$|\rho_{2m+1}| \leq \frac{2(m+1)(1-\psi)}{\sigma(\sigma-1)[m+1]^2 + (m+1)\sigma[2m+1]} + \frac{2(1-\psi)}{\sigma[2m+1]}.$$

For one fold case, Corollary 4, yields the following Corollary:

Corollary 5. *Let $f(z) \in \mathcal{M}^{\sigma}_{\Sigma}(\psi)$, $(\sigma \geq 1, 0 \leq \psi < 1, (z,v) \in \Omega)$ be given (1). Then*

$$|\rho_2| \leq \begin{cases} \sqrt{\frac{2(1-\psi)}{\sigma(2\sigma+1)}} & 0 \leq \psi \leq \frac{1}{3}, \\ \frac{(1-\psi)}{\sigma} & \frac{1}{3} \leq \psi < 1, \end{cases}$$

and

$$|\rho_3| \leq \frac{(1-\psi)(2\sigma-3\psi+3)}{3\sigma^2}.$$

Remark 6. Corollary 5 gives above is the improvement of the estimates for coefficients on $|\rho_2|$ and $|\rho_3|$ investigated by Girgaonkar *et al.*, [29].

Corollary 6. [29] *Let $f(z) \in \mathcal{M}^{\sigma}_{\Sigma}(\psi)$, $(\sigma \geq 1, 0 \leq \psi < 1, (z,v) \in \Omega)$ be given (1). Then*

$$|\rho_2| \leq \sqrt{\frac{2(1-\psi)}{\sigma(2\sigma+1)}},$$

and

$$|\rho_3| \leq \frac{(1-\psi)(2\sigma-3\psi+3)}{3\sigma^2}.$$

Taking $\sigma = 1$ in Corollary 7, we get the following result:

Corollary 7. [11] *Let* $f(z) \in \mathcal{M}^{\sigma}_{\Sigma}(\psi)$, $(\sigma \geq 1, 0 \leq \psi < 1, (z, v) \in \Omega)$ *be given* (1). *Then*

$$|\rho_2| \leq \sqrt{\frac{2(1-\psi)}{3}},$$

and

$$|\rho_3| \leq \frac{(1-\psi)(5-3\psi)}{3}.$$

3. Conclusion

In this present paper, two new subclasses indicated by $\mathcal{M}^{q,\sigma}_{\Sigma,m}(\chi)$ and $\mathcal{M}^{q,\sigma}_{\Sigma,m}(\psi)$ of function class of $\mathcal{E}_m$ was obtained and worked on. Also, the estimates coefficients for $|\rho_{m+1}|$ and $|\rho_{2m+1}|$ of functions in these classes are determined.

Conflicts of Interest: "The author declares no conflict of interest."

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