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A variety of uncertainty principles for the Hankel-Stockwell transform

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Abstract: In this work, we establish L^p local uncertainty principle for the Hankel-Stockwell transform and we deduce L^p version of Heisenberg-Pauli-Weyl uncertainty principle. Next, By combining these principles and the techniques of Donoho-Stark we present uncertainty principles of concentration type in the L^p theory, when $1 < p \leqslant 2$. Finally, Pitt's inequality and Beckner's uncertainty principle are proved for this transform.

Keywords: Hankel-Stockwell transform, local uncertainty principles, Heisenberg-Pauli-Weyl inequality, concentration uncertainty principles, Pitt's inequality, Beckner's inequality.

MSC: 42B10, 42C40.

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1. Introduction

In harmonic analysis, uncertainty principles play an important role. It states that a non-zero function and its Fourier transform cannot be simultaneously sharply concentrated. many of them have already been studied from several points of view for the Fourier transform, Heisenberg-Pauli-Weyl inequality [1] and local uncertainty inequality [2]. As a classical uncertainty principle, the Heisenberg uncertainty principle has been extended to transforms such as the spherical mean transforms [3,4], the Dunkl transform [5] and so forth.

The Hankel transform $\mathcal{H}_\alpha$ is defined for every integrable function f on $\mathbb{R}_+ = [0, +\infty[$ with respect to the measure $d\nu_\alpha$, by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(x) j_\alpha(\lambda x) d\nu_\alpha(x),$$

where $d\nu_\alpha$ is the measure defined on $\mathbb{R}_+$ by

$$d\nu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dx,$$

and j_α is the modified Bessel function given in the next section.

The Hankel transform is found as a very useful mathematical tool in many fields of physics, signal processing and other [6,7]. Also, many uncertainty principles related to this transform $\mathcal{H}_\alpha$ have been proved [8–10].

Time-frequency analysis plays an important role in harmonic analysis, in particular in signal theory. With the development of time-frequency analysis, the study of uncertainty principles have gained considerable attention and have been extended to a wide class of integral transforms such as Weinstein transforms [11,12], Dunkl transforms [13], Hankel-Stockwell transforms [14] and so on.

Based on the ideas of Faris [15] and Price [2,16], we show a general form of the local uncertainty principles for the Hankel-Stockwell transform and we deduce L^p version of Heisenberg-Pauli-Weyl uncertainty principle. We shall use also the Heisenberg uncertainty principle, the properties of the Hankel-Stockwell transform and the techniques of Donoho-Stark [17,18], we show a continuous-time principle for the L^p theory, when $1 < p \leqslant 2$. Finally, Pitt's inequality and Beckner's uncertainty principle are proved for this transform.

This work is organized as follows; in Section 2 we recall some harmonic analysis results related to the Hankel transform. In Section 3, we present some elements of harmonic analysis related to the Hankel-Stockwell transform. In Section 4, we introduce some uncertainty principles for this transform.

2. The Hankel transform

In this section, we summarize some harmonic analysis tools related to the Hankel transform that will be used hereafter, (see [19]). The modified Bessel function $x \longmapsto j_\alpha(x)$ has the following integral representation [20,21];

$$j_\alpha(x) = \begin{cases} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(xt)dt, & \text{if } \alpha > \frac{-1}{2}; \\ \cos x, & \text{if } \alpha = \frac{-1}{2}. \end{cases}$$

In particular, for every $x \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$\left| j_\alpha^{(k)}(x) \right| \leqslant 1.$$

We define the Hankel translation operators τ_x, $x \in [0, +\infty[$ by

$$\tau_x(f)(y) = \begin{cases} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(\sqrt{x^2+y^2+2xy\cos\theta}, x+y) \sin^{2\alpha}(\theta)d\theta, & \text{if } \alpha > \frac{-1}{2}, \\ \frac{f(x+y)+f(|x-y|)}{2}, & \text{if } \alpha = \frac{-1}{2}, \end{cases}$$

whenever the integral in the right-hand side is well defined. In the following, we denote by;

• $S_e(\mathbb{R})$ the Schwartz space, constituted by the even infinitely differentiable functions on the real line, rapidly decreasing together with all their derivatives,

• $L^p(d\nu_\alpha)$ the Lebesgue space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{p,\nu_\alpha} < +\infty$.

For every $f \in L^p(d\nu_\alpha)$, $p \in [1, +\infty]$, and for every $x \in \mathbb{R}_+$, the function $\tau_x(f)$ belongs to the space $L^p(d\nu_\alpha)$ and

$$\|\tau_x(f)\|_{p,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}.$$

In particular, for every $x, y \in \mathbb{R}_+$, we have

$$\tau_x(f)(y) = \tau_y(f)(x).$$

If $f \in L^1(d\nu_\alpha)$, then

$$\int_0^{+\infty} \tau_x(f)(y)d\nu_\alpha(y) = \int_0^{+\infty} f(y)d\nu_\alpha(y).$$

The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined by

$$f * g(x) = \int_0^{+\infty} \tau_x(f)(y)g(y)d\nu_\alpha(y).$$

Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for every $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$, the function $f * g$ belongs to the space $L^r(d\nu_\alpha)$, and we have the following Young's inequality

$$\|f * g\|_{r,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}.$$

The Hankel transform $\mathcal{H}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by

$$\forall \lambda \in \mathbb{R}\,;\; \mathcal{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(x) j_\alpha(\lambda x) d\nu_\alpha(x).$$

Theorem 1. (1) **[Inversion formula]** *Let $f \in L^1(d\nu_\alpha)$ such that $\mathcal{H}_\alpha(f) \in L^1(d\nu_\alpha)$, then we have*

$$f(x) = \int_0^{+\infty} \mathcal{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) d\nu_\alpha(\lambda), \text{ a.e.}$$

(2) **[Plancherel theorem]** *The Fourier transform $\mathcal{H}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself and we have*

$$\|\mathcal{H}_\alpha(f)\|_{2,\nu_\alpha} = \|f\|_{2,\nu_\alpha}.$$

(3) **[Parseval's formula]** *For all functions f and g in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty} f(x)\overline{g(x)} d\nu_\alpha(x) = \int_0^{+\infty} \mathcal{H}_\alpha(f)(\lambda)\overline{\mathcal{H}_\alpha(g)(\lambda)} d\nu_\alpha(\lambda).$$

The Hankel transform $\mathcal{H}_\alpha$ satisfies the following properties;

For every $f \in L^1(d\nu_\alpha)$ and $g \in L^p(d\nu_\alpha), p = 1,2$, the function $f * g$ belongs to $L^p(d\nu_\alpha), p = 1,2$, and we have

$$\mathcal{H}_\alpha(f * g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g).$$

Let $f, g \in L^2(d\nu_\alpha)$. Then $f * g \in L^2(d\nu_\alpha)$, if and only if $\mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g) \in L^2(d\nu_\alpha)$ and we have

$$\mathcal{H}_\alpha(f * g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g), \tag{1}$$

moreover,

$$\int_0^{+\infty} |f * g(x)|^2 d\nu_\alpha(x) = \int_0^{+\infty} |\mathcal{H}_\alpha(f)(\lambda)|^2 |\mathcal{H}_\alpha(g)(\lambda)|^2 d\nu_\alpha(\lambda),$$

where both integrals are finite or infinite.

3. Hankel-Stockwell transform

We recall some results introduced and proved in [14]. The modulation operator is defined for every function ψ in $L^2(d\nu_\alpha)$ by

$$M_a(\psi) = \mathcal{H}_\alpha\left(\sqrt{\tau_a(|\mathcal{H}_\alpha(\psi)|^2)}\right), \quad a > 0.$$

Then for every $\psi \in L^2(d\nu_\alpha)$, $M_a(\psi)$ belongs to $L^2(d\nu_\alpha)$ and we have

$$\|M_a(\psi)\|_{2,\nu_\alpha} = \|\psi\|_{2,\nu_\alpha}.$$

Now, for every $\psi \in L^2(d\nu_\alpha)$, we consider the family $\psi^{a,r}, (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+$ defined by

$$\forall x \in \mathbb{R}_+, \quad \psi^{a,r}(x) = \tau_r M_a D_a \psi(x),$$

where D_a is the dilatation operator given by

$$D_a(\psi)(x) = a^{\alpha+1}\psi(ax).$$

Then, we have the following properties;

(i) For every $\psi \in L^2(d\nu_\alpha)$

$$\tau_x D_a(\psi) = D_a \tau_a x(\psi). \tag{2}$$

(ii) For every $\psi \in L^2(d\nu_\alpha)$

$$\mathcal{H}_\alpha(D_a(\psi)) = D_{\frac{1}{a}}(\mathcal{H}_\alpha(\psi)). \tag{3}$$

Definition 1. A nonzero function $\psi \in L^2(d\nu_\alpha)$ is said to be an admissible window function if

$$0 < C_\psi = \frac{1}{2^\alpha\Gamma(\alpha+1)} \int_0^{+\infty} \tau_1(|\mathcal{H}_\alpha(\psi)|^2)(a)\frac{da}{a} < +\infty.$$

In the following we denote by μ_α the measure defined on $\mathbb{R}_+^* \times \mathbb{R}_+$ by

$$d\mu_\alpha(a,r) = d\nu_\alpha(a)d\nu_\alpha(r),$$

and $L^p(d\mu_\alpha), 1 \leqslant p \leqslant +\infty$, the Lebesgue space on $\mathbb{R}_+^* \times \mathbb{R}_+$ with respect to the measure μ_α with the L^p-norm denoted by $\|.\|_{p,\mu_\alpha}$.

Definition 2. Let ψ be an admissible window function. The continuous Hankel-Stockwell transform S_ψ^α is defined in $L^2(d\nu_\alpha)$ by

$$\begin{aligned} S_\psi^\alpha(f)(a,r) &= \int_0^{+\infty} f(s)\overline{\psi^{a,r}(s)}d\nu_\alpha(s) \\ &= f * M_a D_a(\overline{\psi})(r) = f * D_a M_1(\overline{\psi})(r) = \langle f, \psi^{a,r}\rangle_{\nu_\alpha}, \end{aligned} \tag{4}$$

where $\langle , \rangle_{\nu_\alpha}$ is the usual inner product in the Hilbert space $L^2(d\nu_\alpha)$.

Proposition 1. *Let ψ be an admissible window function. For every $f \in L^2(d\nu_\alpha)$, we have*

$$\|S_\psi^\alpha(f)\|_{\infty,\mu_\alpha} \leqslant \|f\|_{2,\nu_\alpha}\|\psi\|_{2,\nu_\alpha}. \tag{5}$$

Proposition 2. *Let ψ be an admissible window function.*

(i) **[Plancherel formula]** *For every $f \in L^2(d\nu_\alpha)$, we have*

$$\|S_\psi^\alpha(f)\|_{2,\mu_\alpha} = \sqrt{C_\psi}\|f\|_{2,\nu_\alpha}. \tag{6}$$

(ii) **[Parseval formula]** *For all $f, h \in L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\overline{S_\psi^\alpha(h)(a,r)}d\mu_\alpha(a,r) = C_\psi\int_0^{+\infty} f(s)\overline{h(s)}d\nu_\alpha(s).$$

(iii) **[Inversion formula]** *For all $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$, such that $\mathcal{H}_\alpha(f)$ belongs to $L^1(d\nu_\alpha)$, we have*

$$f(u) = \frac{1}{C_\psi}\int_0^{+\infty}\left(\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\psi^{a,r}(u)d\nu_\alpha(r)\right)d\nu_\alpha(a), a.e.,$$

where for each $u \in \mathbb{R}_+$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

By Riesz-Thorin's interpolation theorem we obtain the following.

Proposition 3. *Let ψ be an admissible window function, $f \in L^2(d\nu_\alpha)$ and $2 \leqslant p \leqslant +\infty$, then we have*

$$\|S_\psi^\alpha(f)\|_{p,\mu_\alpha} \leqslant C_\psi^{\frac{1}{p}}\|\psi\|_{2,\nu_\alpha}^{1-\frac{2}{p}}\|f\|_{2,\nu_\alpha}. \tag{7}$$

4. Uncertainty principle for the Hankel-Stockwell transform

In this section, we obtain some uncertainty principles for the Hankel-Stockwell transform.

Theorem 2. **[L^p local uncertainty principle for S_ψ^α]** *Let ψ be an admissible window function and Σ be measurable subset of $\mathbb{R}_+^* \times \mathbb{R}_+$ such that $0 < \mu_\alpha(\Sigma) < +\infty$. Let $p \in]1,2]$, $q = \frac{p}{p-1}$, then or every $f \in L^p(d\nu_\alpha)$, we have*

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{q,\mu_\alpha} \leqslant \begin{cases} C_1(b,\psi)(\mu_\alpha(\Sigma))^{\frac{b}{\alpha+1}}\left(\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^bf\|_{2,\nu_\alpha}\right), & \text{if } 0<b<\frac{\alpha+1}{q}, \\ C_2(b,\psi)(\mu_\alpha(\Sigma))^{\frac{1}{q}}\|f\|_{2p,\nu_\alpha}^{1-\frac{\alpha+1}{qb}}\|r^bf\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{qb}}, & \text{if } b>\frac{\alpha+1}{q}, \\ C_3(b,\psi)(\mu_\alpha(\Sigma))^{\frac{1}{2q}}\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac12}\|r^bf\|_{2,\nu_\alpha}^{\frac12}+\|f\|_{2p,\nu_\alpha}^{\frac12}\|r^bf\|_{2p,\nu_\alpha}^{\frac12}\right), & \text{if } b=\frac{\alpha+1}{q}. \end{cases}$$

where

$$C_1(b,\psi)=\left(\frac{1}{2^{\alpha+1}\Gamma(\alpha+1)(\alpha+1-bq)}\right)^{\frac{b}{2(\alpha+1)}}C_\psi^{\frac1q-\frac{b}{\alpha+1}}\|\psi\|_{2,\nu_\alpha},$$

$$C_2(b,\psi)=\left(\frac{\Gamma(\frac{\alpha+1}{bp})\Gamma\left(\frac{qb-(\alpha+1)}{bp}\right)}{bp2^{\alpha+1}\Gamma(\alpha+1)\Gamma\left(\frac{q}{p}\right)}\right)^{\frac{1}{2q}}\left(\frac{qb}{qb-(\alpha+1)}\right)^{\frac{1}{2p}}\left(\frac{qb}{\alpha+1}-1\right)^{\frac{\alpha+1}{2qbp}}\|\psi\|_{2,\nu_\alpha}, \quad \textit{and}$$

$$C_3(b,\psi)=2C_1\left(\frac{b}{2},\psi\right).$$

Proof. (i) It is clear that the first inequality holds if

$$\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^bf\|_{2,\nu_\alpha}=+\infty.$$

Let $f\in L^p(d\nu_\alpha), 1<p\leqslant 2, q=\frac{p}{p-1}$ such that

$$\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^bf\|_{2,\nu_\alpha}<+\infty.$$

Denote by χ_Σ the characteristic function associated to Σ. Using Minkowski's inequality, relations (5) and (7), we obtain for every $\rho>0$

$$\begin{aligned}\|\chi_\Sigma S_\psi^\alpha(f)\|_{q,\mu_\alpha} &\leqslant \|\chi_\Sigma S_\psi^\alpha(\chi_{[0,\rho[}f)\|_{q,\mu_\alpha}+\|\chi_\Sigma S_\psi^\alpha(f)(\chi_{[\rho,+\infty[}f)\|_{q,\mu_\alpha}\\ &\leqslant (\mu_\alpha(\Sigma))^{\frac1q}\|S_\psi^\alpha(\chi_{[0,\rho[}f)\|_{\infty,\mu_\alpha}+\|S_\psi^\alpha(f)(\chi_{[\rho,+\infty[}f)\|_{q,\mu_\alpha}\\ &\leqslant (\mu_\alpha(\Sigma))^{\frac1q}\|\psi\|_{2,\nu_\alpha}\|\chi_{[0,\rho[}f\|_{2,\nu_\alpha}+C_\psi^{\frac1q}\|\psi\|_{2,\nu_\alpha}^{1-\frac2q}\|\chi_{[\rho,+\infty[}f\|_{2,\nu_\alpha}.\end{aligned}$$

On the other hand, by Hölder's inequality

$$\|\chi_{[0,\rho[}f\|_{2,\nu_\alpha}\leqslant\|r^{-b}\chi_{[0,\rho[}\|_{2q,\nu_\alpha}\|r^bf\|_{2p,\nu_\alpha}.$$

By simple calculus and the hypothesis $0<b<\frac{\alpha+1}{q}$, we obtain

$$\|\chi_{[0,\rho[}f\|_{2,\nu_\alpha}\leqslant C_{b,\alpha,q}\rho^{\frac{\alpha+1}{q}-b}\|r^bf\|_{2p,\nu_\alpha}, \tag{8}$$

where $C_{b,\alpha,q}=\left(\frac{1}{2^{\alpha+1}\Gamma(\alpha+1)(\alpha+1-bq)}\right)^{\frac{1}{2q}}$. Moreover,

$$\|\chi_{[\rho,+\infty[}f\|_{2,\nu_\alpha}\leqslant\rho^{-b}\|r^bf\|_{2,\nu_\alpha}. \tag{9}$$

From (8) and (9), we get

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{q,\mu_\alpha}\leqslant C_{b,\alpha,q}(\mu_\alpha(\Sigma))^{\frac1q}\|\psi\|_{2,\nu_\alpha}\rho^{\frac{\alpha+1}{q}-b}\|r^bf\|_{2p,\nu_\alpha}+\rho^{-b}C_\psi^{\frac1q}\|\psi\|_{2,\nu_\alpha}^{1-\frac2q}\|r^bf\|_{2,\nu_\alpha}.$$

We choose

$$\rho=(C_{b,\alpha,q})^{\frac{-q}{\alpha+1}}(\mu_\alpha(\Sigma))^{\frac{-1}{\alpha+1}}C_\psi^{\frac{1}{\alpha+1}},$$

hence, we obtain the first inequality.

(ii) It is clear that the second inequality holds if $\|f\|_{2p,\nu_\alpha}$ or $\|r^b f\|_{2p,\nu_\alpha} = +\infty$. Assume that

$$\|f\|_{2p,\nu_\alpha} + \|r^b f\|_{2p,\nu_\alpha} < +\infty.$$

From hypothesis $b > \frac{\alpha+1}{q}$, we deduce that the function $r \longrightarrow (1+r^{2bp})^{\frac{-1}{p}}$ belongs to $L^q(d\nu_\alpha)$ and By Hölder's inequality, we have

$$\begin{aligned}\|f\|_{2,\nu_\alpha}^{2p} &= \left(\int_0^{+\infty}(1+r^{2bp})^{\frac{-1}{p}}(1+r^{2bp})^{\frac{1}{p}}|f(r)|^2 d\nu_\alpha(r)\right)^p \\ &\leqslant \left(\int_0^{+\infty}\frac{d\nu_\alpha(r)}{(1+r^{2bp})^{\frac{q}{p}}}\right)^{\frac{p}{q}}\left(\|f\|_{2p,\nu_\alpha}^{2p} + \|r^b f\|_{2p,\nu_\alpha}^{2p}\right). \end{aligned} \tag{10}$$

However, with a standard computation, we obtain

$$\left(\int_0^{+\infty}\frac{d\nu_\alpha(r)}{(1+r^{2bp})^{\frac{q}{p}}}\right)^{\frac{p}{q}} = \left(\frac{\Gamma(\frac{\alpha+1}{bp})\Gamma(\frac{qb-(\alpha+1)}{bp})}{bp2^{\alpha+1}\Gamma(\alpha+1)\Gamma(\frac{q}{p})}\right)^{\frac{p}{q}} = M_{b,\alpha,q}^{\frac{p}{q}}.$$

Replacing $f(r)$ by $f_t(r) = f(rt), t > 0$ in the relation (10), we deduce that for all $t > 0$

$$\|f\|_{2,\nu_\alpha}^{2p} \leqslant M_{b,\alpha,q}^{\frac{p}{q}}\left(t^{(2\alpha+2)(p-1)}\|f\|_{2p,\nu_\alpha}^{2p} + t^{(2\alpha+2)(p-1)-2pb}\|r^b f\|_{2p,\nu_\alpha}^{2p}\right).$$

In particular for $t = \left(\frac{(2bp-(2\alpha+2)(p-1))\|r^b f\|_{2p,\nu_\alpha}^{2p}}{(2\alpha+2)(p-1)\|f\|_{2p,\nu_\alpha}^{2p}}\right)^{\frac{1}{2bp}}$, we obtain

$$\|f\|_{2,\nu_\alpha} \leqslant M_{b,\alpha,q}^{\frac{1}{2q}}\left(\frac{qb}{qb-(\alpha+1)}\right)^{\frac{1}{2p}}\left(\frac{qb}{\alpha+1}-1\right)^{\frac{\alpha+1}{2qbp}}\|f\|_{2p,\nu_\alpha}^{1-\frac{\alpha+1}{qb}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{qb}}.$$

Moreover,

$$\begin{aligned}\|\chi_\Sigma S_\psi^\alpha(f)\|_{q,\mu_\alpha} &\leqslant (\mu_\alpha(\Sigma))^{\frac{1}{q}}\|S_\psi^\alpha(f)\|_{\infty,\mu_\alpha} \\ &\leqslant (\mu_\alpha(\Sigma))^{\frac{1}{q}}\|f\|_{2,\nu_\alpha}\|\psi\|_{2,\nu_\alpha} \\ &\leqslant M_{b,\alpha,q}^{\frac{1}{2q}}\left(\frac{qb}{qb-(\alpha+1)}\right)^{\frac{1}{2p}}\left(\frac{qb}{\alpha+1}-1\right)^{\frac{\alpha+1}{2qbp}}(\mu_\alpha(\Sigma))^{\frac{1}{q}}\|f\|_{2p,\nu_\alpha}^{1-\frac{\alpha+1}{qb}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{qb}}\|\psi\|_{2,\nu_\alpha}.\end{aligned}$$

This completes the proof of the second inequality.

(iii) Let $s > 0$, from the inequality

$$\left(\frac{r}{s}\right)^{\frac{\alpha+1}{2q}} \leqslant 1 + \left(\frac{r}{s}\right)^{\frac{\alpha+1}{q}},$$

it follows that

$$\|r^{\frac{\alpha+1}{2q}} f\|_{2p,\nu_\alpha} \leqslant s^{\frac{\alpha+1}{2q}}\|f\|_{2p,\nu_\alpha} + s^{\frac{-(\alpha+1)}{2q}}\|r^{\frac{\alpha+1}{q}} f\|_{2p,\nu_\alpha}.$$

In particular, by choosing $s = \|r^{\frac{\alpha+1}{q}} f\|_{2p,\nu_\alpha}^{\frac{q}{\alpha+1}}\|f\|_{2p,\nu_\alpha}^{\frac{-q}{\alpha+1}}$, we obtain

$$\|r^{\frac{\alpha+1}{2q}} f\|_{2p,\nu_\alpha} \leqslant 2\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^{\frac{\alpha+1}{q}} f\|_{2p,\nu_\alpha}^{\frac{1}{2}}.$$

Similarly, we prove that

$$\|r^{\frac{\alpha+1}{2q}} f\|_{2,\nu_\alpha} \leqslant 2\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^{\frac{\alpha+1}{q}} f\|_{2,\nu_\alpha}^{\frac{1}{2}}.$$

Thus, we deduce that

$$\begin{aligned}\|\chi_\Sigma S^\alpha_\psi(f)\|_{q,\mu_\alpha} &\leqslant C_1(\frac{\alpha+1}{2q},\psi)(\mu_\alpha(\Sigma))^{\frac{1}{2q}}\left(\|r^{\frac{\alpha+1}{2q}}f\|_{2p,\nu_\alpha}+\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|r^{\frac{\alpha+1}{2q}}f\|_{2,\nu_\alpha}\right)\\ &\leqslant 2C_1(\frac{\alpha+1}{2q},\psi)(\mu_\alpha(\Sigma))^{\frac{1}{2q}}\left(\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|f\|^{\frac12}_{2,\nu_\alpha}\|r^{\frac{\alpha+1}{q}}f\|^{\frac12}_{2,\nu_\alpha}+\|f\|^{\frac12}_{2p,\nu_\alpha}\|r^{\frac{\alpha+1}{q}}f\|^{\frac12}_{2p,\nu_\alpha}\right),\end{aligned}$$

which gives the result for $b=\frac{\alpha+1}{q}$.
□

From the L^p local uncertainty principle, we can find the following L^p Heisenberg-Pauli-Weyl uncertainty principle for the Hankel-Stockwell transform.

Theorem 3. **[L^p Heisenberg-Pauli-Weyl uncertainty principle for the Hankel-Stockwell transform]** *Let ψ be an admissible window function, $p\in]1,2]$, $q=\frac{p}{p-1}$, and $d>0$. Then for every $f\in L^p(d\nu_\alpha)$, we have*

$$\|S^\alpha_\psi(f)\|_{q,\mu_\alpha}\leqslant\begin{cases} C_1(b,d,\psi)\left(\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|r^bf\|_{2,\nu_\alpha}\right)^{\frac{d}{d+4b}}\||(a,r)|^dS^\alpha_\psi(f)\|^{\frac{4b}{d+4b}}_{q,\mu_\alpha}, & \text{if } 0<b<\frac{\alpha+1}{q},\\ C_2(b,d,\psi)\|f\|^{\frac{d}{4\alpha+4+dq}(q-\frac{\alpha+1}{b})}_{2p,\nu_\alpha}\|r^bf\|^{\frac{d(\alpha+1)}{b(4\alpha+4+dq)}}_{2p,\nu_\alpha}\||(a,r)|^dS^\alpha_\psi(f)\|^{\frac{4\alpha+4}{4\alpha+4+dq}}_{q,\mu_\alpha}, & \text{if } b>\frac{\alpha+1}{q},\\ C_3(b,d,\psi)\left(\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|f\|^{\frac12}_{2,\nu_\alpha}\|r^bf\|^{\frac12}_{2,\nu_\alpha}+\|f\|^{\frac12}_{2p,\nu_\alpha}\|r^bf\|^{\frac12}_{2p,\nu_\alpha}\right)^{\frac{d}{2b+d}}\||(a,r)|^dS^\alpha_\psi(f)\|^{\frac{2b}{2b+d}}_{q,\mu_\alpha}, & \text{if } b=\frac{\alpha+1}{q},\end{cases}$$

where

$$C_1(b,d,\psi)=\frac{(C_1(b,\psi))^{\frac{d}{d+4b}}}{\left(2^{2\alpha+2}\Gamma(2\alpha+3)\right)^{\frac{db}{(\alpha+1)(d+4b)}}}\left(\left(\frac{d}{4b}\right)^{\frac{4b}{d+4b}}+\left(\frac{4b}{d}\right)^{\frac{d}{d+4b}}\right)^{\frac1q},$$

$$C_2(b,d,\psi)=\frac{(C_2(b,\psi))^{\frac{dq}{4\alpha+4+dq}}}{\left(2^{2\alpha+2}\Gamma(2\alpha+3)\right)^{\frac{d}{4\alpha+4+dq}}}\left(\left(\frac{dq}{4\alpha+4}\right)^{\frac{4\alpha+4}{4\alpha+4+dq}}+\left(\frac{4\alpha+4}{dq}\right)^{\frac{dq}{4\alpha+4+dq}}\right)^{\frac1q},\quad \textit{and}$$

$$C_3(b,d,\psi)=\frac{(C_3(b,\psi))^{\frac{d}{d+2b}}}{\left(2^{2\alpha+2}\Gamma(2\alpha+3)\right)^{\frac{d}{2q(d+2b)}}}\left(\left(\frac{d}{2b}\right)^{\frac{2b}{d+2b}}+\left(\frac{2b}{d}\right)^{\frac{d}{d+2b}}\right)^{\frac1q}.$$

Proof. (i) Let $0<b<\frac{\alpha+1}{q}, d>0$. For $\rho>0$, let $\widetilde{B}_\rho=\{(a,r)\in\mathbb{R}^*_+\times\mathbb{R}_+;\ a^2+r^2\leqslant\rho^2\}$. Then

$$\|S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}=\|\chi_{\widetilde{B}_\rho}S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}+\|\chi_{\widetilde{B}^c_\rho}S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}. \tag{11}$$

From Theorem 2, we get

$$\|\chi_{\widetilde{B}_\rho}S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}\leqslant C^q_1(b,\psi)(\mu_\alpha(\widetilde{B}_\rho))^{\frac{bq}{\alpha+1}}\left(\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|r^bf\|_{2,\nu_\alpha}\right)^q.$$

On the other hand,

$$\mu_\alpha(\widetilde{B}_\rho)=\frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)}.$$

Using the previous result, we obtain

$$\|\chi_{\widetilde{B}_\rho}S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}\leqslant C^q_1(b,\psi)\left(\frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)}\right)^{\frac{bq}{\alpha+1}}\left(\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|^{-\frac{2}{q}}_{2,\nu_\alpha}\|r^bf\|_{2,\nu_\alpha}\right)^q. \tag{12}$$

Moreover,

$$\|\chi_{\widetilde{B}^c_\rho}S^\alpha_\psi(f)\|^q_{q,\mu_\alpha}\leqslant\rho^{-dq}\||(a,r)|^dS^\alpha_\psi(f)\|^q_{q,\mu_\alpha}. \tag{13}$$

By Combining relations (11), (12) and (13), we get

$$\|S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q \leqslant C_1^q(b,\psi)(\frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)})^{\frac{bq}{\alpha+1}}\left(\|r^b f\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^b f\|_{2,\nu_\alpha}\right)^q+\rho^{-dq}\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q.$$

We choose

$$\rho=\left(\frac{d(2^{2\alpha+2}\Gamma(2\alpha+3))^{\frac{bq}{\alpha+1}}}{4bC_1^q(b,\psi)}\right)^{\frac{1}{(d+4b)q}}\left(\frac{\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}}{\|r^b f\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^b f\|_{2,\nu_\alpha}}\right)^{\frac{1}{d+4b}},$$

hence, we obtain the first inequality.

(ii) Let $b>\frac{\alpha+1}{q}$, $d>0$ and let $\rho>0$. From Theorem 2, we obtain

$$\begin{aligned}\|\chi_{\widetilde{B}_\rho}S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q &\leqslant C_2^q(b,\psi)\mu_\alpha(\widetilde{B}_\rho)\|f\|_{2p,\nu_\alpha}^{q-\frac{\alpha+1}{b}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{b}}\\ &=C_2^q(b,\psi)\frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)}\|f\|_{2p,\nu_\alpha}^{q-\frac{\alpha+1}{b}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{b}}.\end{aligned}\tag{14}$$

Combining the relations (11), (13) and (14), we get

$$\|S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q \leqslant C_2^q(b,\psi)\frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\Gamma(2\alpha+3)}\|f\|_{2p,\nu_\alpha}^{q-\frac{\alpha+1}{b}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{b}}+\rho^{-dq}\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q.$$

We choose

$$\rho=\left(\frac{dq2^{2\alpha+2}\Gamma(2\alpha+3)\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q}{(4\alpha+4)C_2^q(b,\psi)\|f\|_{2p,\nu_\alpha}^{q-\frac{\alpha+1}{b}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{\alpha+1}{b}}}\right)^{\frac{1}{4\alpha+4+dq}},$$

hence, we obtain the second inequality.

(iii) Let $b=\frac{\alpha+1}{q}$, $d>0$ and let $\rho>0$. From Theorem 2, we get

$$\begin{aligned}\|\chi_{\widetilde{B}_\rho}S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q &\leqslant C_3^q(b,\psi)(\mu_\alpha(\widetilde{B}_\rho))^{\frac{1}{2}}\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2,\nu_\alpha}^{\frac{1}{2}}+\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\right)^q\\ &=C_3^q(b,\psi)\frac{\rho^{2\alpha+2}}{\sqrt{2^{2\alpha+2}\Gamma(2\alpha+3)}}\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2,\nu_\alpha}^{\frac{1}{2}}+\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\right)^q.\end{aligned}\tag{15}$$

Combining the relations (11), (13) and (15), we obtain

$$\begin{aligned}\|S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q \leqslant & C_3^q(b,\psi)\frac{\rho^{2\alpha+2}}{\sqrt{2^{2\alpha+2}\Gamma(2\alpha+3)}}\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2,\nu_\alpha}^{\frac{1}{2}}+\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\right)^q\\ &+\rho^{-dq}\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}^q.\end{aligned}$$

We choose

$$\rho=\left(\frac{dq(2^{2\alpha+2}\Gamma(2\alpha+3))^{\frac{1}{2}}}{C_3^q(b,\psi)(2\alpha+2)}\right)^{\frac{1}{2\alpha+2+dq}}\times\left(\frac{\||(a,r)|^d S_\psi^\alpha(f)\|_{q,\mu_\alpha}}{\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2,\nu_\alpha}^{\frac{1}{2}}+\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^b f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\right)}\right)^{\frac{q}{2\alpha+2+dq}},$$

hence, we obtain the result.

□

In the following, we shall use the L^p Heisenberg-Pauli-Weyl uncertainty principle to obtain a concentration uncertainty principle.

Definition 3. Let $0 \leqslant \varepsilon < 1$ and let S be a measurable set of $\mathbb{R}_+$. We say that $f \in L^p(d\nu_\alpha)$, $p \in [1,2]$, is ε-concentrated on S in $L^p(d\nu_\alpha)$-norm if there is a measurable function h vanishing outside S such that

$$\|f-h\|_{p,\nu_\alpha} \leqslant \varepsilon \|f\|_{p,\nu_\alpha}.$$

We introduce a projection operator P_S as $P_S f(r) = f(r), \quad \text{if} \quad r \in S$ and $P_S f(r) = 0, \quad \text{if} \quad r \notin S$. Let $0 \leqslant \varepsilon_S < 1$. Then f is ε_S-concentrated on S in $L^p(d\nu_\alpha)$-norm if and only if

$$\|f - P_S f\|_{p,\nu_\alpha} \leqslant \varepsilon_S \|f\|_{p,\nu_\alpha}.$$

Definition 4. Let ψ be an admissible window function and Σ be a measurable set of $\mathbb{R}_+^* \times \mathbb{R}_+$. We define a projection operator Q_Σ as

$$Q_\Sigma f = (S_\psi^\alpha)^{-1}\Big(P_\Sigma(S_\psi^\alpha(f))\Big).$$

Let $0 \leqslant \varepsilon_\Sigma < 1$. Then S_ψ^α is ε_Σ -concentrated on Σ in $L^q(d\mu_\alpha)$-norm, $1 \leqslant q \leqslant 2$ if and only if

$$\|S_\psi^\alpha(f) - S_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha} \leqslant \varepsilon_\Sigma \|S_\psi^\alpha(f)\|_{q,\mu_\alpha}.$$

Proposition 4. *Let ψ be an admissible window function and Σ be a measurable set of $\mathbb{R}_+^* \times \mathbb{R}_+$. Then, for every $p > 2$ and $\varepsilon > 0$, if S_ψ^α is ε-concentrated in Σ with respect to the norm $\|.\|_{2,\mu_\alpha}$, then*

$$(\mu_\alpha(\Sigma))^{\frac{p-2}{p}} \geqslant (1-\varepsilon^2) C_\psi^{1-\frac{2}{p}} \|\psi\|_{2,\nu_\alpha}^{\frac{4}{p}-2},$$

where $\mu_\alpha(\Sigma) = \displaystyle\int\int_\Sigma d\nu_\alpha(a) d\nu_\alpha(r)$.

Proof. Let $f \in L^2(d\nu_\alpha)$ and $p > 2$. As $S_\psi^\alpha(f)$ is ε-concentrated in Σ with respect to the norm $\|.\|_{2,\mu_\alpha}$, we have

$$\|\chi_{\Sigma^c} S_\psi^\alpha(f)\|_{2,\mu_\alpha} \leqslant \varepsilon \sqrt{C_\psi} \|f\|_{2,\nu_\alpha}.$$

Now, using relation (6), we get

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 \geqslant (1-\varepsilon^2) C_\psi \|f\|_{2,\nu_\alpha}^2.$$

Applying Hölder's inequality, we obtain

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 \leqslant \|S_\psi^\alpha(f)\|_{p,\mu_\alpha}^2 (\mu_\alpha(\Sigma))^{\frac{p-2}{p}}.$$

By relation (7), we obtain

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 \leqslant C_\psi^{\frac{2}{p}} \|\psi\|_{2,\nu_\alpha}^{2-\frac{4}{p}} \|f\|_{2,\nu_\alpha}^2 (\mu_\alpha(\Sigma))^{\frac{p-2}{p}}.$$

Finally,

$$(\mu_\alpha(\Sigma))^{\frac{p-2}{p}} \geqslant (1-\varepsilon^2) C_\psi^{1-\frac{2}{p}} \|\psi\|_{2,\nu_\alpha}^{\frac{4}{p}-2}.$$

□

Proposition 5. *Let ψ be an admissible window function and $f \in L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ such that $\|S_\psi^\alpha(f)\|_{2,\mu_\alpha} = 1$. If f is ε_S-concentrated on S in $L^1(d\nu_\alpha)$-norm and $S_\psi^\alpha(f)$ is ε_Σ-concentrated on Σ in $L^2(d\mu_\alpha)$-norm, then $\nu_\alpha(S) \geqslant C_\psi(1-\varepsilon_S)^2 \|f\|_{1,\nu_\alpha}^2$, and $\mu_\alpha(\Sigma)\|f\|_{2,\nu_\alpha}^2 \|\psi\|_{2,\nu_\alpha}^2 \geqslant 1 - \varepsilon_\Sigma^2$.*

Proof. As $S_\psi^\alpha(f)$ is ε_Σ-concentrated on Σ in $L^2(d\mu_\alpha)$-norm and by the orthogonality of the projection operator P_Σ, it follows that

$$\|S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 - \|S_\psi^\alpha(f) - P_\Sigma(S_\psi^\alpha(f))\|_{2,\mu_\alpha}^2 = \|P_\Sigma(S_\psi^\alpha(f))\|_{2,\mu_\alpha}^2 \geqslant 1 - \varepsilon_\Sigma^2,$$

and thus

$$1-\varepsilon_{\Sigma}^{2}\leqslant\|S_{\psi}^{\alpha}(f)\|_{\infty,\mu_\alpha}^{2}\mu_\alpha(\Sigma)\leqslant\mu_\alpha(\Sigma)\|f\|_{2,\nu_\alpha}^{2}\|\psi\|_{2,\nu_\alpha}^{2}.$$

By the same way, f is ε_S-concentrated on S in $L^1(d\nu_\alpha)$-norm, we obtain

$$(1-\varepsilon_S)\|f\|_{1,\nu_\alpha}\leqslant\int_S|f(r)|d\nu_\alpha(r).$$

Now, by the Cauchy-Schwarz inequality and the fact that $\|f\|_{2,\nu_\alpha}=\dfrac{1}{\sqrt{C_\psi}}$, we get

$$(1-\varepsilon_S)\|f\|_{1,\nu_\alpha}\leqslant\frac{\nu_\alpha^{\frac{1}{2}}(S)}{\sqrt{C_\psi}}.$$

□

Definition 5. Let ψ be an admissible window function and Σ be a measurable set of $\mathbb{R}_+^*\times\mathbb{R}_+$. Let $d>0$, $f\in L^p(d\nu_\alpha)$, $p\in[1,2]$ and $0\leqslant\varepsilon_\Sigma<1$. We say that $|(a,r)|^dS_\psi^\alpha$ is ε_Σ -concentrated on Σ in $L^q(d\mu_\alpha)$-norm, if and only if

$$\||(a,r)|^dS_\psi^\alpha(f)-|(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha}\leqslant\varepsilon_\Sigma\||(a,r)|^dS_\psi^\alpha(f)\|_{q,\mu_\alpha}.$$

Theorem 4. *Let ψ be an admissible window function and Σ be a measurable set of $\mathbb{R}_+^*\times\mathbb{R}_+$. Let $f\in L^p(d\nu_\alpha)$, $p\in]1,2]$, $0\leqslant\varepsilon_\Sigma<1$ and $d>0$. If $|(a,r)|^dS_\psi^\alpha$ is ε_Σ -concentrated on Σ in $L^q(d\mu_\alpha)$-norm, then*

$$\|S_\psi^\alpha(f)\|_{q,\mu_\alpha}\leqslant\begin{cases}C_1(b,d,\psi)\left(\|r^bf\|_{2p,\nu_\alpha}+\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|r^bf\|_{2,\nu_\alpha}\right)^{\frac{d}{d+4b}}\\ \times\left(\frac{1}{1-\varepsilon_\Sigma}\||(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha}\right)^{\frac{4b}{d+4b}}, & \text{if } 0<b<\frac{\alpha+1}{q},\\ C_2(b,d,\psi)\|f\|_{2p,\nu_\alpha}^{\frac{d}{4\alpha+4+dq}(q-\frac{\alpha+1}{b})}\|r^bf\|_{2p,\nu_\alpha}^{\frac{d(\alpha+1)}{b(4\alpha+4+dq)}}\left(\frac{1}{1-\varepsilon_\Sigma}\||(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha}\right)^{\frac{4\alpha+4}{4\alpha+4+dq}}, & \text{if } b>\frac{\alpha+1}{q},\\ C_3(b,d,\psi)\left(\|\psi\|_{2,\nu_\alpha}^{-\frac{2}{q}}\|f\|_{2,\nu_\alpha}^{\frac{1}{2}}\|r^bf\|_{2,\nu_\alpha}^{\frac{1}{2}}+\|f\|_{2p,\nu_\alpha}^{\frac{1}{2}}\|r^bf\|_{2p,\nu_\alpha}^{\frac{1}{2}}\right)^{\frac{d}{2b+d}}\\ \times\left(\frac{1}{1-\varepsilon_\Sigma}\||(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha}\right)^{\frac{2b}{2b+d}}, & \text{if } b=\frac{\alpha+1}{q}.\end{cases}$$

Proof. Let $f\in L^p(d\nu_\alpha)$, $p\in]1,2]$. Since $|(a,r)|^dS_\psi^\alpha$ is ε_Σ -concentrated on Σ in $L^q(d\mu_\alpha)$-norm, then we have

$$\||(a,r)|^dS_\psi^\alpha(f)\|_{q,\mu_\alpha}\leqslant\||(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha}+\varepsilon_\Sigma\||(a,r)|^dS_\psi^\alpha(f)\|_{q,\mu_\alpha}.$$

Thus,

$$\||(a,r)|^dS_\psi^\alpha(f)\|_{q,\mu_\alpha}\leqslant\frac{1}{1-\varepsilon_\Sigma}\||(a,r)|^dS_\psi^\alpha(Q_\Sigma f)\|_{q,\mu_\alpha},$$

and we obtain the result from Theorem 3. □

Definition 6. Let Σ be a measurable set of $\mathbb{R}_+^*\times\mathbb{R}_+$ and $0\leqslant\eta<\sqrt{C_\psi}$. Then a nonzero function $f\in L^p(d\nu_\alpha), 1\leqslant p\leqslant 2$ is η-bandlimited on Σ in $L^q(d\mu_\alpha)$-norm, if

$$\|\chi_{\Sigma^c}S_\psi^\alpha(f)\|_{q,\mu_\alpha}\leqslant\eta\|f\|_{p,\nu_\alpha}.$$

where $q=\frac{p}{p-1}$.

Corollary 1. *Let ψ be an admissible window function.*

(i) *If* $0 < b < \frac{\alpha+1}{2}$, *then there exists a positive constant* C *such that for every function* f *which is* η*-bandlimited on* Σ

$$(\mu_\alpha(\Sigma))^{\frac{2b}{\alpha+1}}\left(\|r^b f\|_{4,\nu_\alpha} + \|\psi\|_{2,\nu_\alpha}^{-1}\|r^b f\|_{2,\nu_\alpha}\right)^2 \geqslant C(C_\psi - \eta^2)\|f\|_{2,\nu_\alpha}^2.$$

(ii) *If* $b > \frac{\alpha+1}{2}$, *then there exists a positive constant* C *such that for every function* f *which is* η*-bandlimited on* Σ

$$\mu_\alpha(\Sigma)\|f\|_{4,\nu_\alpha}^{2-\frac{\alpha+1}{b}}\|r^b f\|_{4,\nu_\alpha}^{\frac{\alpha+1}{b}} \geqslant C(C_\psi - \eta^2)\|f\|_{2,\nu_\alpha}^2.$$

Proof. Since $f \in L^2(d\nu_\alpha)$ is η-bandlimited on Σ, then

$$\|\chi_\Sigma S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 = C_\psi\|f\|_{2,\nu_\alpha}^2 - \|\chi_{\Sigma^c} S_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 \geqslant (C_\psi - \eta^2)\|f\|_{2,\nu_\alpha}^2.$$

For (i) and (ii), we use the local inequalities given respectively by Theorem 2. □

According to the following Pitt's inequality for the Hankel transform [9], we obtain the Pitt's inequality for the Hankel-Stockwell transform.

Proposition 6. *Let* $0 \leqslant \eta < \alpha + 1$. *For every* $f \in S_e(\mathbb{R})$, *we have*

$$\int_0^{+\infty} |\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2 d\nu_\alpha(\lambda) \leqslant C_{\eta,\alpha}\int_0^{+\infty}|r|^\eta|f(r)|^2 d\nu_\alpha(r), \tag{16}$$

where $C_{\eta,\alpha} = 2^{-\eta}\left(\dfrac{\Gamma\left(\frac{2\alpha+2-\eta}{4}\right)}{\Gamma\left(\frac{2\alpha+2+\eta}{4}\right)}\right)^2$ *and* $\Gamma(.)$ *denotes the well known Eurler's gamma function.*

Theorem 5. [Pitt's inequality the Hankel- Stockwell transform] *Let* ψ *be an admissible window function and* $0 \leqslant \eta < \alpha + 1$. *For every* $f \in S_e(\mathbb{R})$, *the Pitt's inequality for the Hankel- Stockwell transform is given by*

$$C_\psi\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2 d\nu_\alpha(\lambda) \leqslant C_{\eta,\alpha}\int_0^{+\infty}\int_0^{+\infty}|r|^\eta|S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r).$$

Proof. For $\eta = 0$, the result follows from relation (6). Now suppose that $0 < \eta < \alpha + 1$. For every $f \in S_e(\mathbb{R})$ and by (16), we can write

$$\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(\lambda)|^2 d\nu_\alpha(\lambda) \leqslant C_{\eta,\alpha}\int_0^{+\infty}|r|^\eta|S_\psi^\alpha(f)(a,r)|^2 d\nu_\alpha(r).$$

Integrating with respect $d\nu_\alpha(a)$, we get

$$C_{\eta,\alpha}\int_0^{+\infty}\int_0^{+\infty}|r|^\eta|S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r) \geqslant \int_0^{+\infty}\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(\lambda)|^2 d\nu_\alpha(a)d\nu_\alpha(\lambda). \tag{17}$$

By (1)–(4) and using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(\lambda)|^2 d\nu_\alpha(a)d\nu_\alpha(\lambda)\\
&= \int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2\Big(\int_0^{+\infty}|\mathcal{H}_\alpha(D_a M_1(\psi))(\lambda)|^2 d\nu_\alpha(a)\Big)d\nu_\alpha(\lambda)\\
&= \int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2\Big(\int_0^{+\infty}|D_{\frac{1}{a}}\left(\sqrt{\tau_1(|\mathcal{H}_\alpha(\psi)|^2)}\right)(\lambda)|^2 d\nu_\alpha(a)\Big)d\nu_\alpha(\lambda)\\
&= \frac{1}{a^{\alpha+1}}\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2\Big(\int_0^{+\infty}D_{\frac{1}{a}}\left(\tau_1(|\mathcal{H}_\alpha(\psi)|^2)(\lambda)\right)d\nu_\alpha(a)\Big)d\nu_\alpha(\lambda)\\
&= \frac{1}{a^{2\alpha+2}}\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2\Big(\int_0^{+\infty}\tau_1(|\mathcal{H}_\alpha(\psi)|^2)(\frac{\lambda}{a})d\nu_\alpha(a)\Big)d\nu_\alpha(\lambda)\\
&= C_\psi\int_0^{+\infty}|\lambda|^{-\eta}|\mathcal{H}_\alpha(f)(\lambda)|^2 d\nu_\alpha(\lambda).
\end{aligned} \tag{18}$$

Relations (17) and (18) gives the Pitt's inequality for the Hankel-Stockwell transform. □

Now, using the following logarithmic uncertainty principle for the Hankel transform [9], we obtain the logarithmic uncertainty principle for the Hankel-Stockwell transform.

Proposition 7. *For every $f \in S_e(\mathbb{R})$, the following inequality holds:*

$$\int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(f)(t)|^2 d\nu_\alpha(t) + \int_0^{+\infty} \ln(r)|f(r)|^2 d\nu_\alpha(r) \geqslant \Big(\ln 2 + \omega(\frac{\alpha+1}{2})\Big) \int_0^{+\infty} |f(r)|^2 d\nu_\alpha(r), \tag{19}$$

where ω denotes the logarithmic derivative of the gamma function Γ [20,21].

Theorem 6. [Logarithmic uncertainty principle for the Hankel-Stockwell transform] *Let ψ be an admissible window function. For every $f \in S_e(\mathbb{R})$, we have*

$$C_\psi \int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(f)(t)|^2 d\nu_\alpha(t) + \int_0^{+\infty}\int_0^{+\infty} \ln(r)|S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r) \geqslant C_\psi\Big(\ln 2 + \omega(\frac{\alpha+1}{2})\Big)\|f\|^2_{2,\nu_\alpha}.$$

Proof. Replacing f by $S_\psi^\alpha(f)$ in the inequality (19), we obtain

$$\begin{aligned}\int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(t)|^2 d\nu_\alpha(t) + \int_0^{+\infty} \ln(r)|S_\psi^\alpha(f)(a,r)|^2 d\nu_\alpha(r)\\ \geqslant \left(\ln 2 + \omega\left(\frac{\alpha+1}{2}\right)\right) \int_0^{+\infty} |S_\psi^\alpha(f)(a,r)|^2 d\nu_\alpha(r).\end{aligned}$$

Integrating both sides with respect to a, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(t)|^2 d\mu_\alpha(a,t) + \int_0^{+\infty}\int_0^{+\infty} \ln(r)|S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r)\\ \geqslant \left(\ln 2 + \omega\left(\frac{\alpha+1}{2}\right)\right) \int_0^{+\infty}\int_0^{+\infty} |S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r).\end{aligned} \tag{20}$$

By (1), (4) and using Fubini's theorem, we obtain

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} &\ln(t)|\mathcal{H}_\alpha(S_\psi^\alpha(f)(a,.))(t)|^2 d\mu_\alpha(a,t)\\ &= \int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(f)(t)|^2\Big(\int_0^{+\infty} |\mathcal{H}_\alpha(D_a M_1(\psi))(t)|^2 d\nu_\alpha(a)\Big) d\nu_\alpha(t)\\ &= C_\psi \int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(f)(t)|^2 d\nu_\alpha(t).\end{aligned} \tag{21}$$

Hence, by (6), (20) and (21), we have

$$C_\psi \int_0^{+\infty} \ln(t)|\mathcal{H}_\alpha(f)(t)|^2 d\nu_\alpha(t) + \int_0^{+\infty}\int_0^{+\infty} \ln(r)|S_\psi^\alpha(f)(a,r)|^2 d\mu_\alpha(a,r) \geqslant C_\psi \left(\ln 2 + \omega\left(\frac{\alpha+1}{2}\right)\right) \|f\|^2_{2,\nu_\alpha}.$$

□

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