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Covering radius of repetition codes over $F_2 + vF_2 + v^2F_2$ with $v^3 = 1$

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Abstract: In this paper, the exact value of covering radius of unit repetition codes and the bounds of covering radius of zero-divisor repetition codes have been determined by using Lee weight over the finite ring $F_2 + vF_2 + v^2F_2$. Moreover the covering radius of different block repetition codes have been also studied.

Keywords: Covering radius, finite ring, unit repetition codes, zero-divisor repetition codes.

MSC: 11H71, 94B05, 68P30.

1. Introduction

For more than a decade, codes over finite rings have gotten much attention of researchers due the definition of Gray map [1–3]. Particularly, codes over the ring $F_2 + vF_2 + v^2F_2$ have been extensively studied [4,5].

The covering radius is very interesting topic in coding theory. First time, the covering radius of binary linear codes were studied by Helleseth *et al.*, [6]. Furthermore, covering radius of linear codes over $F_2 + uF_2$ with $u^2 = 0$ was determined by using Lee distance, Chinese Euclidean distance, and Bachoc distance [7–9].

Recently, the covering radius of codes over Z_4 have been determined by using lee weight and Chinese Euclidean lee weight [10,11]. In [11], *Manoj et al.*, introduced new reduction and torsion codes, repetition codes for octonary codes and determined its covering radius. Chatouh *et al.*, [12], Pandian *et al.*, [13] gave the upper and lower bounds on the covering radius of some classes of codes over rings $Z_2 \times Z_4$, $Z_2 \times R$ with $R = F2 + vF2, v^2 = v$ respectively. Panchanathan *et al.*, in [14] studied bounds on covering radius for various repetition codes with respect to different and similar length over $F_2 + uF_2 + u^2F_2$ with $u^3 = 0$ using Lee weight and generalized Lee weight.

The goal of this paper is to investigate the covering radius of repetition codes over the finite ring $F_2 + vF_2 + v^2F_2$ with $v^3 = 1$.

2. Preliminaries

2.1. The overview of the ring $M = F_2 + uF_2 + u^2F_2$ with $u^3 = 1$

Some basic information about the finite ring M have been recalled as follows:

Let $F_2 = \{0, 1\}$ be the binary Galois field of two element. Further, let M is commutative ring with characteristic 2 and 8 elements which is given by $M = \{0, 1, v, v^2, 1 + v, 1 + v^2, v + v^2, 1 + v + v^2 | v^3 = 1\}$ [4]. The elements $1, v, v^2$ are units of M and $\{0, 1 + v, 1 + v^2, v + v^2, 1 + v + v^2\}$ is the set of Zero-divisors of M.

The M-submodule C of M^n is called a linear code having length n over M, and its elements are known as codewords. The Hamming weight $w_H(c)$ of a non-zero codeword $c = (c_1, c_2, \cdots, c_n)$ of C is the number of non-zero coordinates of element c, and the Hamming distance between two codewords x and $y \in M^n$ which is denoted by $d_H(x, y)$ is equal to $w_H(x - y)$.

For any $m = a + bv + cv^2 \in M$, the Gray map g from M to F_2^3 is defined as $g(m) = (a + b + c, a + b, a + c)$. The Lee weight w_L for an element $m \in M$ is given by $w_L(m) = w_H(g(m))$. Now, the Lee weight of the element of M are given as:

$$\begin{cases} w_L(0) = 0, w_L(1) = 3, \\ w_L(v) = w_L(v^2) = w_L(v+v^2) = 2, \text{ and} \\ w_L(1+v) = w_L(1+v^2) = w_L(1+v+v^2) = 1. \end{cases}$$

Lee weight of a codeword $c = (c_1, c_2, \cdots, c_n)$ of a linear code C is defined as $w_L(c) = \sum_{i=1}^{n} w_L(c_i)$, and Lee distance between two codewords m_1, m_2 of C, (with $(m_1, m_2) \neq 0$) is denoted by $d_L(m_1, m_2)$ and defined as $d_L(m_1, m_2) = w_L(m_1 - m_2)$.

2.2. Covering radius of linear codes

Definition 1. Let C be a linear code over M of length n. The covering radius of C with respect distance d, where $d \in \{d_H, d_L\}$, is defined as:

$$r_d(C) = max\{d(x, C) | x \in M^n\}.$$

Proposition 1. *[15] Let C be a code generated by G over M, C' be a code generated by G' over M and C'' be a code generated by $\begin{pmatrix} 0 & G' \\ G'' & A \end{pmatrix}$, then $r_d(C'') \leqslant r_d(C) + r_d(C'')$. Moreover, if B be the concatenation of C and C' then $r_d(B) \geqslant r_d(C) + r_d(C'')$.*

3. Main results

3.1. Covering radius of repetition codes

In this section we represents the covering radius of repetition code over F_q, as well as over M.

Let F_q be a finite field of q element. Let C be a repetition code over F_q of length n defined by $C = \{(a, a, \cdots, a) | a \in F_q\}$ with G its generator matrix $G = \begin{pmatrix} a & a & \cdots & a \end{pmatrix}$. Hence C is $[n, 1, n]$-code over M, and according to [16], the covering radius of C is equal to $\left\lfloor \frac{n(q-1)}{q} \right\rfloor$, and the covering radius of $[n(q-1), 1, n(q-1)]$-codes over F_q with generator matrix $\begin{pmatrix} 1, 1, \cdots, 1 | & a_2, a_2, \cdots, a_2 | & \cdots | & a_{q-1}, a_{q-1}, \cdots, a_{q-1} \end{pmatrix}$ is equal to $\left\lfloor \frac{n(q-1)^2}{q} \right\rfloor$.

Theorem 1. *Let C_1 be unit repetition code of generator matrix $G_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$ and C_2 be zero-divisor repetition code with generator matrix $G_2 = \begin{pmatrix} 1+v & 1+v & \cdots & 1+v \end{pmatrix}$ and C_3 be zero-divisor repetition code with generator matrix $G_3 = \begin{pmatrix} 1+v+v^2 & 1+v+v^2 & \cdots & 1+v+v^2 \end{pmatrix}$. Then we have*

i) C_1 is $[n, 8, d_H = n, d_L = 3n]$-code over M and $r_{d_L}(C_1) = \frac{3n}{2}$,
ii) C_2 is $[n, 4, d_H = n, d_L = n]$-code over M and $4\left\lfloor \frac{n}{4} \right\rfloor \leq r_{d_L}(C_2) \leq 2n$,
iii) C_3 is $[n, 2, d_H = n, d_L = n]$-code over M and $\lfloor \frac{n}{2} \rfloor \leq r_{d_L}(C_3) \leq \frac{5n}{2}$.

Proof. i) Since $C_1 = \{a.(1, 1, \cdots, 1) | a \in M\}$ is linear code of length n and 8 elements, $d_H(C_1) = min\{w_H(c) | c \in C, c \neq (0, 0, \cdots, 0)\} = n$ and $d_L(C_1) = nw_L(1) = 3n$. So C_1 is $[n, |C_1| = 8, n, 3n]$-code over M. Now, we will prove that $r_{d_L}(C_1) = \frac{3n}{2}$.

Let $x \in M^n$, and

$$\begin{cases} w_0 \text{ be the number of coordinate has 0 component,} \\ w_1 \text{ the number of coordinate has 1 component,} \\ w_2 \text{ the number of coordinate has } v \text{ component,} \\ w_3 \text{ is the number of coordinate has } v^2 \text{ component,} \\ w_4 \text{ the number of coordinate has } 1+v \text{ component,} \\ w_5 \text{ the number of coordinate has } 1+v^2 \text{ component,} \\ w_6 \text{ is the number of coordinate has } v+v^2 \text{ component, and} \\ w_7 \text{ be the number of coordinate has } 1+v+v^2 \text{ component.} \end{cases}$$

We also have

$$\sum_{i=0}^{7} w_i = n$$

and

$$\begin{cases} w_L(x) = w_0w_L(0) + w_1w_L(1) + w_2w_L(v) + w_3w_L(v^2) + w_4w_L(1+v) + w_5w_L(1+v^2) + w_6w_L(v+v^2) \\ \qquad +w_7w_L(1+v+v^2) = (w_4+w_5+w_7) + 2(w_2+w_3+w_6) + 3w_1. \\ d_L(x, 00...0) = (w_4+w_5+w_7) + 2(w_2+w_3+w_6) + 3w_1 = n - w_0 + 2w_1 + w_2 + w_3 + w_6, \\ d_L(x, 11...1) = w_L(x - 11...1) = n + 2w_0 - w_1 + w_4 + w_5 + w_7. \end{cases}$$

And by the same way, we have

$$\begin{cases} d_L(x, vv...v) = n + w_0 - w_2 + w_3 + 2w_4 + w_6, \\ d_L(x, v^2v^2...v^2) = n + w_0 + w_2 - w_3 + 2w_5 + w_6, \\ d_L(x, 1+v1+v...1+v) = n + w_1 + 2w_2 - w_4 + w_5 + w_7, \\ d_L(x, 1+v^21+v^2...1+v^2) = n + w_1 + 2w_3 + +w_4 - w_5 + w_7, \\ d_L(x, v+v^2v+v^2...v+v^2) = n + w_0 + w_2 + w_3 - w_6 + 2w_7, \\ d_L(x, 1+v+v^21+v+v^2...1+v+v^2) = n + w_1 + w_4 + w_5 + 2w_6 - w_7. \end{cases}$$

So,

$$d_L(x, C_1) = min\{d_L(x,y)|y \in C_1\} \leqslant \frac{1}{8}(8n+4n) = \frac{12n}{8} = \frac{3n}{2}$$

implies

$$d_L(x, C_1) \le \frac{3n}{2} \ \ \forall x \in M^n,$$

hence

$$r_{d_L}(C_1) \le \frac{3n}{2}.$$

To prove the reverse inequality, take

$$x = \Big[\overbrace{00\cdots 0}^{m}|\overbrace{11\cdots 1}^{m}|\overbrace{vv\cdots v}^{m}|\overbrace{v^2v^2\cdots v^2}^{m}|\overbrace{1+v1+v\cdots 1+v}^{m}|\overbrace{1+v^21+v^2\cdots 1+v^2}^{m} \\ |\overbrace{v+v^2v+v^2...v+v^2}^{m}|\overbrace{1+v+v^21+v+v^2\cdots 1+v+v^2}^{mn-7m} \Big] \in M^n, \text{ with } m = \lfloor \tfrac{n}{8} \rfloor.$$

So, we have

$$\begin{cases} d_L(x, 00\cdots 0) = 2n - 4m, \\ d_L(x, 11\cdots 1) = n + 4m, \\ d_L(x, vv\cdots v) = n + 4m, \\ d_L(x, v^2v^2\cdots v^2) = n + 4m, \\ d_L(x, v+1v+1\cdots v+1) = 2n - 4m, \\ d_L(x, v^2+1v^2+1\cdots v^2+1) = 2n - 4m, \\ d_L(x, v+v^2v+v^2\cdots v+v^2) = 3n - 12m, \\ d_L(x, 1+v+v^21+v+v^2\cdots 1+v+v^2) = n + 4m, \\ d_L(x, C_1) = min\{n+4m, 2n-4m, 3n-12m\} = n + 4m. \end{cases}$$

Then $r_{d_L}(C_1) \ge n + 4m \ge n + \frac{n}{2} = \frac{3n}{2}$. Hence, $r_{d_L}(C_1) = \frac{3n}{2}$.

ii) Since $C_2 = \{b.(1+v, 1+v, \cdots, 1+v)|b \in M\} = \{(0,0,\cdots,0), (1+v, 1+v, \cdots, 1+v), (1+v^2, 1+v^2, \cdots, 1+v^2), (v+v^2, v+v^2, \cdots, v+v^2)\}$, $d_H(C_2) = n$ and $d_L(C_2) = n$. So C_2 is $[n, |C_2| = 4, n, n]$-code over M.

Now, we prove $4\left\lfloor \frac{n}{4} \right\rfloor \le r_{d_L}(C_2) \le 2n$. To prove $r_{d_L}(C_2) \le 2n$, suppose $x \in M^n$ then

$$\begin{cases} d_L(x,(0,0,\cdots,0)) = n - w_0 + 2w_1 + w_2 + w_3 + w_6, \\ d_L(x,(1+v,1+v,\cdots,1+v)) = n + w_1 + 2w_2 - w_4 + w_5 + w_7, \\ d_L(x,(1+v^2,1+v^2,\cdots,1+v^2)) = n + w_1 + 2w_3 + w_4 - w_5 + w_7, \text{ and} \\ d_L(x,(v+v^2,v+v^2,\cdots,v+v^2)) = n + w_0 + w_2 + w_3 - w_6 + 2w_7. \end{cases}$$

Now, since $w_1 + w_2 + w_3 + w_7 \leq n$, so $d_L(x, C_2) = min\{n - w_0 + 2w_1 + w_2 + w_3 + w_6, n + w_1 + 2w_2 - w_4 + w_5 + w_7, n + w_1 + 2w_3 + w_4 - w_5 + w_7, n + w_0 + w_2 + w_3 - w_6 + 2w_7\} \leq \frac{1}{4}(4n + 4(w_1 + w_2 + w_3 + w_7)) \leq 2n$.

To prove $4\left\lfloor \frac{n}{4} \right\rfloor \leq r_{d_L}(C_2)$, suppose

$$x = \left[\overbrace{00\cdots0}^{\lfloor \frac{n}{4} \rfloor} | \overbrace{1+v1+v\cdots1+v}^{\lfloor \frac{n}{4} \rfloor} | \overbrace{1+v^21+v^2\cdots1+v^2}^{\lfloor \frac{n}{4} \rfloor} | \overbrace{v+v^2v+v^2\cdots v+v^2}^{n-3\lfloor \frac{n}{4} \rfloor} \right] \in M^n,$$

then

$$\begin{cases} d_L(x,(0,0,\cdots,0)) = 2n - 4\lfloor \frac{n}{4} \rfloor, \\ d_L(x,(1+v,1+v,\cdots,1+v)) = d_L(x,(1+v^2,1+v^2,\cdots,1+v^2)) = n, \text{ and} \\ d_L(x,(v+v^2,v+v^2,\cdots,v+v^2)) = 4\lfloor \frac{n}{4} \rfloor. \end{cases}$$

Then $d_L(x, C_2) = min\{n, 2n - 4\lfloor \frac{n}{4} \rfloor, 4\lfloor \frac{n}{4} \rfloor\} = 4\lfloor \frac{n}{4} \rfloor$ implies $r_{d_L}(C_2) \geq 4\lfloor \frac{n}{4} \rfloor$. Hence we achieved our desired result.

iii) Since C_3 is linear code with generator matrix $G_3 = \begin{pmatrix} 1+v+v^2 & 1+v+v^2 & \cdots & 1+v+v^2 \end{pmatrix}$, then $C_3 = \{c.(1+v+v^2, 1+v+v^2, \cdots, 1+v+v^2)/c \in M\} = \{(0,0,\cdots,0), (1+v+v^2, 1+v+v^2, \cdots, 1+v+v^2)\}$, $d_H(C_3) = n$, and $d_L(C_3) = n.w_L(1+v+v^2) = n$. Hence C_3 is $[n, 2, d_H = n, d_L = n]$.

Now we prove that $\lfloor \frac{n}{2} \rfloor \leq r_{d_L}(C_3) \leq \frac{5n}{2}$. To prove $r_{d_L}(C_3) \leq \frac{5n}{2}$, suppose $x \in M^n$ then

$$\begin{cases} d_L(x,(0,0,\cdots,0)) = n - w_0 + 2w_1 + w_2 + w_3 + w_6, \text{ and} \\ d_L(x,(1+v+v^2,1+v+v^2,\cdots,1+v+v^2)) = n + w_1 + w_4 + w_5 + 2w_6 - w_7. \end{cases}$$

Now, because $w_0 + w_1 + w_6 + w_7 \leq n$, so $d_L(x, C_3) \leq \frac{1}{2}(3n - 2w_0 + 2w_1 + 2w_6 - 2w_7)$ implies $d_L(x, C_3) \leq \frac{5n}{2}$ $\forall x \in M^n$. Hence $r_{d_{L(C_3)}} \leq \frac{5n}{2}$.

It remains to prove $\lfloor \frac{n}{2} \rfloor \leq r_{d_L}(C_3)$. For this suppose $x = [\overbrace{00\cdots0}^{\lfloor \frac{n}{2} \rfloor} | \overbrace{1+v+v^21+v+v^2\cdots1+v+v^2}^{\lfloor \frac{n}{2} \rfloor}]$. Then

$$\begin{cases} d_L(x,(0,0,\cdots,0)) = \lfloor \frac{n}{2} \rfloor \text{ and} \\ d_L(x,(1+v+v^2,1+v+v^2,\cdots,1+v+v^2)) = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

So $d_L(x, C_3) = \lfloor \frac{n}{2} \rfloor$ and by the definition of covering radius of code we have $r_{d_{L(C_3)}} \geq \lfloor \frac{n}{2} \rfloor$.
□

3.2. Covering radius of block repetition code over M

Theorem 2. *Let $Rp^{3n}_{U_s}$ be a block repetition code of length $3n$ with generator matrix* $G_{U_s} = \begin{pmatrix} \overbrace{11\cdots1}^{n} & \overbrace{vv\cdots v}^{n} & \overbrace{v^2v^2\cdots v^2}^{n} \end{pmatrix}$ *and $Rp^{n_1+n_2+n_3}_{U_d}$ be a block repetition code of length $n_1 + n_2 + n_3$ with generator matrix* $G_{U_d} = \begin{pmatrix} Vn_1\overbrace{11\cdots1} & \overbrace{vv\cdots v}^{n_2} & \overbrace{v^2v^2\cdots v^2}^{n_3} \end{pmatrix}$. *Then*

i) the code $Rp^{3n}_{U_s}$ is $[3n, 8, d_H = 3n, d_L = 3n]$-code over M with $r_{d_L}(Rp^{3n}_{U_s}) = \frac{9n}{2}$, and
ii) the code $Rp^{n_1+n_2+n_3}_{U_d}$ is $[n_1 + n_2 + n_3, 8, d_H = n_1 + n_2 + n_3, d_L = n_1 + n_2 + n_3]$-code over M with $r_{d_L}(Rp^{n_1+n_2+n_3}_{U_d}) = \frac{3}{2}(n_1 + n_2 + n_3)$.

Proof. i) Let $x \in M^3n$ such that $x = (x_1, x_2, x_3)$ with $x_i, 1 \leq i \leq 3$ is vector of n coordinates with (s_0, t_0, w_0) is the number of coordinate have 0 element in (x_1, x_2, x_3) respectively, and

$$\begin{cases}(s_1,t_1,w_1) \text{ is } 1 \text{ times in } x,\\ (s_2,t_2,w_2) \text{ is } v \text{ times in } x,\\ (s_3,t_3,w_3) \text{ is } v^2 \text{ times in } x,\\ (s_4,t_4,w_4) \text{ is } 1+v \text{ times in } x,\\ (s_5,t_5,w_5) \text{ is } 1+v^2 \text{ times in } x,\\ (s_6,t_6,w_6) \text{ is } v+v^2 \text{ times in } x, \text{ and}\\ (s_7,t_7,w_7) \text{ is } 1+v+v^2 \text{ times in } x.\end{cases}$$

Then we have

$$\sum_{i=0}^{7} s_i = \sum_{j=0}^{7} t_j = \sum_{k=0}^{7} w_i = n.$$

Let $c_{i,i=0\cdots7} \in \{\alpha \left(11\cdots1 \quad vv\cdots v \quad v^2v^2\cdots v^2\right)$ such that $\alpha \in M\}$. Then we have

$$\begin{cases}d_L(x,c_0) = 3n - s_0 + 2s_1 + s_2 + s_3 + s_6 - t_0 + 2t_1 + t_2 + t_3 + t_6 - w_0 + 2w_1 + w_2 + w_3 + w_6,\\ d_L(x,c_1) = 3n + 2s_0 - s_1 + s_4 + s_5 + s_7 + t_0 - t_2 + t_3 + 2t_4 + t_6 + w_0 + w_2 - w_3 + 2w_5 + w_6,\\ d_L(x,c_2) = 3n + s_0 - s_2 + s_3 + 2s_4 + s_6 + t_0 + t_2 - t_3 + 2t_5 + t_6 + 2w_0 - w_1 + w_4 + w_5 + w_7,\\ d_L(x,c_3) = 3n + 2s_0 - s_1 + s_4 + s_5 + s_7 + t_0 - t_2 + t_3 + 2t_4 + t_6 + w_0 + w_2 - w_3 + 2w_5 + w_6,\\ d_L(x,c_4) = 3n + s_1 + 2s_2 - s_4 + s_5 + s_7 + t_0 + t_2 + t_3 - t_6 + 2t_7 + w_1 + 2w_3 + w_4 - w_5 + w_7,\\ d_L(x,c_5) = 3n + s_1 + 2s_3 + s_4 - s_5 + s_7 + t_1 + 2t_2 - t_4 + t_5 + t_7 + w_0 + w_2 + w_3 - w_6 + 2w_7,\\ d_L(x,c_6) = 3n + s_0 + s_2 + s_3 - s_6 + 2s_7 + t_1 + 2t_3 + t_4 - t_5 + t_7 + w_1 + 2w_2 - w_4 + w_5 + w_7,\\ d_L(x,c_7) = 3n + s_1 + s_4 + s_5 + 2s_6 - s_7 + t_1 + t_4 + t_5 + 2t_6 - t_7 + w_1 + 2w_2 + w_5 + 2w_6 - w_7.\end{cases}$$

Therefore $d_L(x, Rp^{3n}_{U_s}) = min\{d_L(x,c_i), i = 0..7\} \leq \frac{1}{8}(24n + 4(\sum_{i=0}^{7} s_i) + 4(\sum_{i=0}^{7} t_i) + 4(\sum_{i=0}^{7} w_i))$ and hence $d_L(x, Rp^{3n}_{U_s}) \leq \frac{9n}{2}\ \forall x \in M^{3n}$.

In other hand by using Preposition 1 we have $r_{d_L}(Rp^{3n}_{U_s}) \geq 3r_{d_L}(C_1) = \dfrac{9n}{2}$.

ii) The prove is similar to **i)** and left for the readers.

□

Theorem 3. *Let C be a block repetition code of length $2n$ generated by $G = \left(1+v\overbrace{1+v\cdots}^{n}1+v \quad 1+v+v^2\overbrace{1+v+v^2}^{n}\cdots1+v+v^2\right)$. Then the code C is $[2n, 8, d_H = n, d_L = n]$-code over M^{2n} and $4\lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor \leq r_{d_L}(C) \leq 4n$.*

Proof. By Proposition 1, we have $r_{d_L} \geq r_{d_L}(C_2) + r_{d_L}(C_3) \geq 4\lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$. So $r_{d_L} \geq 4\lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$.

Let $x \in M^{2n}$ such that $x = (x_1|x_2)$ with $m_i, i = 0..7$ is the number of coordinates have $(0,1,v,v^2,1+v,1+v^2,v+v^2,1+v+v^2)$ respectively in x_1 and $l_i, i = 0..7$ is the number of coordinates in x_2 have $(0,1,v,v^2,1+v,1+v^2,v+v^2,1+v+v^2)$ respectively. We conclude that $\sum_{i=0}^{7} m_i = \sum_{i=0}^{7} l_i = n$.

Now let $c_{i,i=0..7} \in \{\alpha \left(1+v1+v\cdots1+v \quad 1+v+v^21+v+v^2\cdots1+v+v^2\right)$ such that $\alpha \in M\}$, then we have

$$\begin{cases}d_L(x,c_0) = 2n - m_0 + 2m_1 + m_2 + m_3 + m_6 - l_0 + 2l_1 + l_2 + l_3 + l_6,\\ d_L(x,c_1) = 2n + m_1 + 2m_2 - m_4 + m_5 + m_7 + l_1 + l_4 + l_5 + 2l_6 - l_7,\\ d_L(x,c_2) = 2n + m_0 + m_2 + m_3 - m_6 + 2m_7 + l_1 + l_4 + l_5 + 2l_6 - l_7,\\ d_L(x,c_3) = 2n + m_1 + 2m_3 + m_4 - m_5 + m_7 + l_1 + l_4 + l_5 + 2l_6 - l_7,\\ d_L(x,c_4) = 2n + m_1 + 2m_3 + m_4 - m_5 + m_7 - l_0 + 2l_1 + l_2 + l_3 + l_6,\\ d_L(x,c_5) = 2n + m_0 + m_2 + m_3 - m_6 + 2m_7 - l_0 + 2l_1 + l_2 + l_3 + l_6,\\ d_L(x,c_6) = 2n + m_1 + 2m_2 - m_4 + m_5 + m_7 - l_0 + 2l_1 + l_2 + l_3 + l_6,\\ d_L(x,c_7) = 2n - m_0 + 2m_1 + m_2 + m_3 + m_6 + l_1 + l_4 + l_5 + 2l_6 - l_7.\end{cases}$$

By definition of Lee distance of C, we have $d_L(x,C) = min\{d_L(x,c_i), i = 0..7\} \leq 4n$ for any x in M^{2n} which implies that $r_{d_L}(C) \leq 4n$. Hence the prove is complete. □

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