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The Existence and Nonexistence of Entire Positive Radial Solutions of Quasilinear Elliptic Systems with Gradient Term

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Research Article

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Abstract

We study the existence and nonexistence of entire positive solutions for quasilinear elliptic system with gradient term

$$\text{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^{p-1} = a(|x|)f(u,v),$$
$$\text{div}(|\nabla v|^{q-2}\nabla v) + |\nabla v|^{q-1} = b(|x|)g(u,v)$$

on $\mathbf{R}^{\mathbf{N}}(N \geq 3)$, where nonlinearities f and g are positive and continuous, the potentials a and b are continuous, c-positive and satisfy appropriate growth conditions at infinity. We have that entire large positive solutions fail to exist if f and g are sublinear and a and b have fast decay at infinity, while if f and g satisfy some growth conditions at infinity, and a, b are of slow decay or fast decay at infinity, then the system has infinitely many entire solutions, which are large or bounded.

Keywords: Quasilinear elliptic equations; Large solutions; Bounded solution; Entire radial solution.
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1 Introduction

Existence and nonexistence of a quasilinear elliptic system

$$\begin{cases} \text{div}(|\nabla u|^{p-2}\nabla u) + f(x,u,v) = 0, & x \in \mathbf{R}^N, \\ \text{div}(|\nabla v|^{q-2}\nabla v) + g(x,u,v) = 0, & x \in \mathbf{R}^N. \end{cases} \tag{1.1}$$

have been studied by several authors. See, for example, Ph.Clement, R.Manasevich and E.Mitidieri [2], P.L.Felmer, R.Manasevich and F.de Thelin [5], Z.M.Guo [6], E.Mitidieri, G.Sweers and R.vander Vorst [10], Z.D.Yang and Q.S.Lu [16, 18], A.Ben Dkhil and N. Zeddini [25], D.-P. Covei [28-29, 31] and the references therein. Problem (1) arises in the theory of quasi-regular and quasi-conformal mappings as well as in the study of non- Newtonian fluids. In the latter case, the pair (p, q) is a characteristic of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with

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$(p,q) < (2,2)$ are called pseudo-plastics. If $(p,q) = (2,2)$, they are Newtonian fluids.

When $p = q = 2$, the system

$$\begin{cases} \Delta u + f(x,u,v) = 0, & \text{in } \Omega, \\ \Delta v + g(x,u,v) = 0, & \text{in } \Omega. \end{cases}$$

have received much attention recently. We list here, for example, [1,4,7-9, 21-24, 26-27, 30] and refer to the references therein.

When $p = q = 2, f = -a(|x|)v^{\alpha}, g = -b(|x|)u^{\beta}$, system (1.1) becomes

$$\begin{cases} \Delta u = a(|x|)v^{\alpha}, & x \in \mathbf{R}^N, \\ \Delta v = b(|x|)u^{\beta}, & x \in \mathbf{R}^N \end{cases}$$

for which existence results for boundary blow-up positive solutions can be found in a recent paper by Lair and Wood [15]. The authors established that all positive entire radial solutions of systems above are boundary blow-up provided that

$$\int_0^{\infty} ta(t)dt = \infty, \quad \int_0^{\infty} tb(t)dt = \infty.$$

On the other hand, if

$$\int_0^{\infty} ta(t)dt < \infty, \quad \int_0^{\infty} tb(t)dt < \infty.$$

then all positive entire radial solutions of this system are bounded.

F. C$\grave{i}$rstea and V.R$\breve{a}$dulescu [4] extended the above results to a larger class of systems

$$\begin{cases} \Delta u = a(|x|)g(v), & x \in \mathbf{R}^N, \\ \Delta v = b(|x|)f(u), & x \in \mathbf{R}^N \end{cases}$$

Z.D.Yang [17] extended the above results to a class of systems

$$\begin{cases} \text{div}(|\nabla u|^{p-2}\nabla u) = a(|x|)g(v), & x \in \mathbf{R}^N, \\ \text{div}(|\nabla v|^{q-2}\nabla v) = b(|x|)f(u), & x \in \mathbf{R}^N \end{cases}$$

Very recently, Xinguang Zhang, Lishan Liu [3] for which the existence and nonexistence results can be obtained to the elliptic system

$$\begin{cases} \Delta u + |\nabla u| = a(|x|)f(u,v), & x \in \mathbf{R}^N, \\ \Delta v + |\nabla v| = b(|x|)g(u,v), & x \in \mathbf{R}^N \end{cases} \tag{1.2}$$

The corresponding equation that leads us to the system (1.2) is

$$\Delta u + |\nabla u|^{\lambda} = a(|x|)f(u), \quad x \in \Omega, \ 0 < \lambda \leq 2.$$

which was treated in [11-13]. Problems of this type arise in stochastic control theory and have been first studied in [14]. The corresponding parabolic equation was consider in [20].

In this paper, we consider the following quasilinear elliptic system

$$\begin{cases} \text{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^{p-1} = a(|x|)f(u,v), & x \in \mathbf{R}^N, \\ \text{div}(|\nabla v|^{q-2}\nabla v) + |\nabla v|^{q-1} = b(|x|)g(u,v), & x \in \mathbf{R}^N \end{cases} \tag{1.3}$$

where $N \geq 3$. Throughout this paper we always assume a, b are c-positive $C(\mathbf{R}^N)$ functions, f, g : $[0,\infty) \times [0,\infty) \to [0,\infty)$ are nonnegative, continuous and nondecreasing functions for f or g.

For convenience we use the following convention:

· A function p is c-positive in a domain $\Omega \subseteq \mathbf{R}^N$ ig p is nonnegative on Ω and satisfies the following: if $x_0 \in \Omega$ and $p(x_0) = 0$, then there exists a domain Ω_0 such taht $x_0 \in \Omega_0 \subset \Omega$ and $p(x) > 0$ for all $x \in \partial\Omega_0$.

· A solution (u, v) of system

$$\text{div}(|\nabla u|^{p-2}\nabla u) = f(x, u, v), \quad \text{div}(|\nabla v|^{p-2}\nabla v) = g(x, u, v) \tag{$*$}$$

is call an entire large solution (or explosive solution) if it is a classical solution of $(*)$ on $\mathbf{R}^N$ and $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$.

Our purpose is to generalize part results in [3]. The main results of the present paper are complement and extend part results in [3,17,19]. Using an argument inspired by Xinguang Zhang, Lishan Liu [3] and Hong Li, Pei Zhang, Zhijun Zhang [19], we obtain the following main results.

Theorem 1. Suppose f and g satisfy

$$\max\left\{\sup_{s+t\geq 1}\frac{f(s,t)}{(s+t)^{m-1}}, \sup_{s+t\geq 1}\frac{g(s,t)}{(s+t)^{m-1}}\right\} < +\infty, \tag{1.4}$$

and a, b satisfy the decay conditions

$$\int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}a(s)ds\right)^{1/(p-1)} dt < \infty, \quad \int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)^{1/(q-1)} dt < \infty \tag{1.5}$$

where $m = \min\{p, q\}$, then problem (1.3) has no positive entire radial large solution.

Remark 1. If $N \geq 3, N > p$, then condition (1.5) of Theorem 1 is replaced by

$$0 < \int_1^\infty r^{\frac{1}{p-1}} a(r)^{\frac{1}{p-1}} dr < \infty, \text{ if } 1 < p \leq 2, \tag{A}$$

$$0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} a(r) dr < \infty, \quad \text{if } p \geq 2; \tag{B}$$

and

$$0 < \int_1^\infty r^{\frac{1}{q-1}} b(r)^{\frac{1}{q-1}} dr < \infty, \text{ if } 1 < q \leq 2, \tag{C}$$

$$0 < \int_1^\infty r^{\frac{(q-2)N+1}{q-1}} b(r) dr < \infty, \quad \text{if } q \geq 2. \tag{D}$$

Let

$$J(r) = \int_0^r (t^{1-N}\int_0^t s^{N-1}\psi(s)ds)^{\frac{1}{p-1}} dt$$

If fact, if $1 < p \leq 2$, by estimating above the integral

$$J(r) \leq C_1 + \int_1^r t^{\frac{1-N}{p-1}} [\int_0^t s^{N-1}\psi(s)ds]^{1/(p-1)} dt.$$

Using the assumption $N \geq 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one,

$$J(r) \leq C_2 + C_3\int_1^r t^{\frac{3-N-p}{p-1}} \int_1^t s^{\frac{N-1}{p-1}}\psi(s)^{\frac{1}{p-1}} dsdt.$$

Computing the above integral, we obtain

$$J(r) \leq C_2 + C_4\int_1^r t^{\frac{1}{p-1}}\psi(t)^{\frac{1}{p-1}} dt.$$

Applying (A) in the integral above we infer that $H_\infty = \lim_{r\to\infty} J(r) < \infty$. On the other hand, if $p \geq 2$, set

$$H(t) = \int_0^t s^{N-1}\psi(s)ds$$

and note that either, $H(t) \leq 1$ for $t > 0$ or $H(t_0) = 1$ for some $t_0 > 0$. In the first case, $H^{\frac{1}{p-1}} \leq 1$, and hence,

$$J(r) = \int_0^r t^{\frac{1-N}{p-1}} H(t)^{\frac{1}{p-1}} dt \leq C_5 + \int_1^r t^{\frac{1-N}{p-1}} dt$$

so that $J(r)$ has a finite limit because $p < N$. In the second case, $H(s)^{\frac{1}{p-1}} \leq H(s)$ for $s \geq s_0$ and hence,

$$J(r) \leq C_6 + \int_1^r t^{\frac{1-N}{p-1}} \int_0^t s^{N-1}\psi(s)dsdt.$$

Estimating and integrating by parts, we obtain

$$\begin{aligned} J(r) &\leq C_6 + \frac{p-1}{N-p}\int_0^1 t^{N-1}\psi(t)dt + \frac{p-1}{N-p}[\int_1^r t^{\frac{(p-2)N+1}{p-1}}\psi(t)dt - r^{\frac{p-N}{p-1}}\int_0^r t^{N-1}\psi(t)dt] \\ &\leq C_7 + C_8 \int_1^r t^{\frac{(p-2)N+1}{p-1}}\psi(t)dt. \end{aligned}$$

By (B), $H_\infty = \lim_{r\to\infty} J(r) < \infty$.

In order to state the existence results, we denote

$$A_1(\infty) := \lim_{r\to\infty} A_1(r), \quad A_1(r) = \int_0^r \left(t^{1-N}\int_0^t s^{N-1}a(s)ds\right)^{1/(p-1)} dt, \quad r \geq 0,$$

$$B_1(\infty) := \lim_{r\to\infty} B_1(r), \quad B_1(r) = \int_0^r \left(t^{1-N}\int_0^t s^{N-1}b(s)ds\right)^{1/(q-1)} dt, \quad r \geq 0;$$

$$A_2(\infty) := \lim_{r\to\infty} A_2(r), \quad A_2(r) = \int_0^r \left(e^{-t}t^{1-N}\int_0^t e^s s^{N-1}a(s)ds\right)^{1/(p-1)} dt, \quad r \geq 0,$$

$$B_2(\infty) := \lim_{r\to\infty} B_1(r), \quad B_2(r) = \int_0^r \left(e^{-t}t^{1-N}\int_0^t e^s s^{N-1}b(s)ds\right)^{1/(q-1)} dt, \quad r \geq 0;$$

and

$$F(\infty) := \lim_{r\to\infty} F(r), \quad F(r) = \int_\alpha^r \frac{ds}{(f(s,s)+g(s,s))^{1/(m_0-1)}}, \quad r \geq \alpha > 0,$$

where m_0 satisfies

$$m_0 = \begin{cases} \min\{p,q\}, & \text{if } f+g \geq 1, \\ \max\{p,q\}, & \text{if } f+g < 1, \end{cases}$$

we see that

$$F'(r) = \frac{1}{(f(r,r)+g(r,r))^{1/(m_0-1)}} > 0, \forall r > \alpha$$

so, F has the inverse function F^{-1} on $[\alpha, \infty)$.

Theorem 2. Assume

$$F(\infty) = \infty.$$

Then the system (1.3) has infinitely many positive entire radial solutions $(u,v) \in C^1([0,\infty))$. Moreover, the following hold:

(i) If $A_1(\infty) < \infty$ and $B_1(\infty) < \infty$, then all positive entire radial solutions of (1.3) are bounded.

(ii) If $A_2(\infty) = \infty = B_2(\infty)$, then $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$, that is all positive entire radial solutions of (1.3) are large.

Theorem 3. If

$$F(\infty) < \infty, \quad A_1(\infty) < \infty, \quad B_1(\infty) < \infty,$$

and there exist $\beta > \alpha$ and $\gamma > \alpha$ such that

$$A_1(\infty) + B_1(\infty) < F(\infty) - F(\beta + \gamma), \tag{1.6}$$

the system (1.3) has a positive radial bounded solution $(u, v) \in C^1([0, \infty))$ satisfying

$$\beta + f^{1/(p-1)}(\beta,\gamma)A_1(r) \le u(r) \le F^{-1}(F(\beta+\gamma) + A_1(r) + B_1(r)), \quad \forall r \ge 0;$$
$$\gamma + g^{1/(q-1)}(\beta,\gamma)B_1(r) \le v(r) \le F^{-1}(F(\beta+\gamma) + A_1(r) + B_1(r)), \quad \forall r \ge 0.$$

Theorem 4. If m_0 is defined as before, then we have

(i) If

$$A_2(\infty) = \infty = B_2(\infty),$$

and

$$\lim_{s\to\infty} \frac{(f(s,s) + g(s,s))^{1/(m_0-1)}}{s} = 0, \tag{1.7}$$

then the system (1.3) has infinitely many positive entire radial large solutions;

(ii) If

$$A_1(\infty) < \infty, \quad B_1(\infty) < \infty,$$

and

$$\sup_{s\ge 0}(f(s,s) + g(s,s))^{1/(m_0-1)} < \infty, \tag{1.8}$$

then the system (1.3) has infinitely many positive entire bounded radial solutions.

2 Proofs of Theorem 1

In this section, we consider the proof of Theorem 1 by contradictions. Assume that the system (1.3) has the positive entire radial large solution (u, v). From (1.3), we know that

$$(e^t t^{N-1}(u')^{p-1}(t))' = e^t t^{N-1} a(t) f(u(t), v(t)), \quad t \ge 0,$$
$$(e^t t^{N-1}(v')^{q-1}(t))' = e^t t^{N-1} b(t) g(u(t), v(t)), \quad t \ge 0.$$

Now we set

$$U(r) = \max_{0\le t\le r} u(t), \quad V(r) = \max_{0\le t\le r} v(t),$$

it is easy to see that U, V are positive and nondecreasing functions. Moreover, we have $U \ge u, V \ge v$ and $U(r), V(r) \to +\infty$ as $r \to +\infty$. It follows from (1.4) that there exists $C > 0$ such that

$$\max\{f(s,t), g(s,t)\} \le C(s+t)^{m-1}, \quad for \quad s + t \ge 1, \tag{2.1}$$

and

$$\max\{f(s,t), g(s,t)\} \le C, \quad for \quad s + t \le 1. \tag{2.2}$$

Then by (2.1) and (2.2), we have

$$\max\{f(s,t), g(s,t)\} \le C(1+s+t)^{m-1}, \quad for \;\; s+t \ge 0. \tag{2.3}$$

Then, from (2.3) we can get

$$f(u(r), v(r)) \le C(1+u(r)+v(r))^{m-1} \le C(1+U(r)+V(r))^{m-1}, \quad for \;\; r \ge 0.$$

So, for all $r \ge r_0 \ge 0$, we obtain

$$\begin{aligned} u(r) &= u(r_0) + \int_{r_0}^{r} \left(e^{-t} t^{1-N} \int_0^t e^s s^{N-1} a(s) f(u(s), v(s)) ds \right)^{1/(p-1)} dt \\ &\le u(r_0) + C \int_{r_0}^{r} \left(e^{-t} t^{1-N} \int_0^t e^s s^{N-1} a(s) (1+U(s)+V(s))^{m-1} ds \right)^{1/(p-1)} dt \\ &\le u(r_0) + C(1+U(r)+V(r))^{\frac{m-1}{p-1}} \int_{r_0}^{r} \left(e^{-t} t^{1-N} \int_0^t e^s s^{N-1} a(s) ds \right)^{1/(p-1)} dt \\ &\le u(r_0) + C(1+U(r)+V(r))^{\frac{m-1}{p-1}} \int_{r_0}^{r} \left(t^{1-N} \int_0^t s^{N-1} a(s) ds \right)^{1/(p-1)} dt \\ &\le u(r_0) + C(1+U(r)+V(r)) \int_{r_0}^{r} \left(t^{1-N} \int_0^t s^{N-1} a(s) ds \right)^{1/(p-1)} dt \end{aligned}$$

where C is a positive constant. Notice that (1.5), we choose $r_0 > 0$ such that

$$\max\left\{ \int_0^\infty \left(t^{1-N} \int_0^t s^{N-1} a(s) ds \right)^{1/(p-1)} dt, \int_0^\infty \left(t^{1-N} \int_0^t s^{N-1} b(s) ds \right)^{1/(q-1)} dt \right\} < \frac{1}{4C}. \tag{2.4}$$

It follows that $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = \infty$, we can find $r_1 \ge r_0$ such that

$$\bar{U}(r) = \max_{r_0 \le t \le r} u(t), \quad \bar{V}(r) = \max_{r_0 \le t \le r} v(t), \qquad \forall r \ge r_1. \tag{2.5}$$

Thus, we have

$$\bar{U}(r) \le u(r_0) + C(1+\bar{U}(r)+\bar{V}(r)) \int_0^\infty \left(t^{1-N} \int_0^t s^{N-1} a(s) ds \right)^{1/(p-1)} dt, \qquad \forall r \ge r_1.$$

By (2.4), we get

$$\bar{U}(r) \le u(r_0) + \frac{(1+\bar{U}(r)+\bar{V}(r))}{4}, \qquad \forall r \ge r_1.$$

that is

$$\bar{U}(r) \le C_1 + \frac{(\bar{U}(r)+\bar{V}(r))}{4}, \qquad \forall r \ge r_1.$$

where $C_1 = \frac{1}{4} + u(r_0) > 0$. Similarly,

$$\bar{V}(r) \le C_2 + \frac{(\bar{U}(r)+\bar{V}(r))}{4}, \qquad \forall r \ge r_1.$$

which implies

$$\bar{U}(r) + \bar{V}(r) \le 2(C_1 + C_2), \qquad \forall r \ge r_1. \tag{2.6}$$

(1.7) means that $\bar{U}$ and $\bar{V}$ are bounded and so u and v are bounded which is a contradiction. It follows that (1.3) has no positive entire radial large solutions and the proof is now completed.

3 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2. We start by showing that (1.3) has positive radial solutions. On this purpose we fix $\beta > \alpha$ and $\gamma > \alpha$ and we show that the system

$$\begin{cases} (\Phi_p(u'))' + \frac{N-1}{r}(\Phi_p(u')) + \Phi_p(u') = a(r)f(u(r), v(r)), \\ (\Phi_q(v'))' + \frac{N-1}{r}(\Phi_q(v')) + \Phi_p(v') = b(r)g(u(r), v(r)), \quad r > 0, \\ u(0) = \beta > 0, \ \ v(0) = \gamma > 0; \ \ u', v' \geq 0, \quad \text{on } [0, \infty), \end{cases} \tag{3.1}$$

has solutions (u, v) (where $\Phi_p(s) = |s|^{p-2}s$). Thus $U(x) = u(|x|), V(x) = v(|x|)$ are positive solutions of (1.3). Integrating (3.1) we have

$$u(r) = \beta + \int_0^r \left(e^{-t}t^{1-N} \int_0^t e^s s^{N-1} a(s) f(u(s), v(s)) ds \right)^{1/(p-1)} dt, \quad r \geq 0,$$

$$v(r) = \gamma + \int_0^r \left(e^{-t}t^{1-N} \int_0^t e^s s^{N-1} b(s) g(u(s), v(s)) ds \right)^{1/(q-1)} dt, \quad r \geq 0.$$

Let $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ be the sequences of positive continuous functions defined on $[0, \infty)$ by

$$\begin{cases} u_0(r) = \beta, v_0(r) = \gamma, \\ u_{n+1}(r) = \beta + \int_0^r \left(e^{-t}t^{1-N} \int_0^t e^s s^{N-1} a(s) f(u_n(s), v_n(s)) ds \right)^{1/(p-1)} dt, \\ v_{n+1}(r) = \gamma + \int_0^r \left(e^{-t}t^{1-N} \int_0^t e^s s^{N-1} b(s) g(u(s), v(s)) ds \right)^{1/(q-1)} dt, \quad r \geq 0, \end{cases} \tag{3.2}$$

Obviously, for all $r \geq 0$, we have

$$u_n(r) \geq \beta, \quad v_n(r) \geq \gamma, \quad u_0 \leq u_1, \quad v_0 \leq v_1.$$

The monotonicity of f and g yield

$$u_1(r) \leq u_2(r), \quad v_1(r) \leq v_2(r), \quad r \geq 0.$$

Repeating such arguments we deduce that

$$u_n(r) \leq u_{n+1}(r), \quad v_n(r) \leq v_{n+1}(r), \quad r \geq 0, \ \ n \geq 1.$$

and we obtain that sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are nondecreasing on $[0, \infty)$. Notice

$$\begin{aligned} u'_{n+1}(r) \ &= \ \left(e^{-r}r^{1-N} \int_0^r e^s s^{N-1} a(s) f(u_n(s), v_n(s)) ds \right)^{1/(p-1)} \\ &\leq \left(f(u_n(r), v_n(r)) \right)^{1/(p-1)} A'_1(r) \\ &\leq \left(f(u_n(r) + v_n(r), u_n(r) + v_n(r) \right)^{1/(p-1)} A'_1(r) \\ &\leq (f(u_{n+1}(r) + v_{n+1}(r), u_{n+1}(r) + v_{n+1}(r)) \\ &\quad + g(u_{n+1}(r) + v_{n+1}(r), u_{n+1}(r) + v_{n+1}(r)))^{1/(p-1)} A'_1(r) \end{aligned}$$

and

$$\begin{aligned} v'_{n+1}(r) \ &= \ \left(e^{-r}r^{1-N} \int_0^r e^s s^{N-1} b(s) g(u_n(s), v_n(s)) ds \right)^{1/(q-1)} \\ &\leq \left(g(u_n(r), v_n(r)) \right)^{1/(q-1)} B'_1(r) \\ &\leq \left(g(u_n(r) + v_n(r), u_n(r) + v_n(r) \right)^{1/(q-1)} B'_1(r) \\ &\leq (f(u_{n+1}(r) + v_{n+1}(r), u_{n+1}(r) + v_{n+1}(r)) \\ &\quad + g(u_{n+1}(r) + v_{n+1}(r), u_{n+1}(r) + v_{n+1}(r)))^{1/(q-1)} B'_1(r) \end{aligned}$$

which implies

$$\frac{u_n'(r)+v_n'(r)}{(f(u_n(r)+v_n(r),u_n(r)+v_n(r))+g(u_n(r)+v_n(r),u_n(r)+v_n(r)))^{1/(m_0-1)}} \le A_1'(r)+B_1'(r).$$

Where m_0 has been defined before. And then integrating on $(0,r)$ we obtain

$$\int_0^r \frac{u_n'(t)+v_n'(t)}{(f(u_n(t)+v_n(t),u_n(t)+v_n(t))+g(u_n(t)+v_n(t),u_n(t)+v_n(t)))^{1/(m_0-1)}}dt \le A_1(r)+B_1(r),$$

So

$$\int_{\beta+\gamma}^{u_n(r)+v_n(r)} \frac{d\tau}{(f(\tau,\tau)+g(\tau,\tau))^{1/(m_0-1)}} \le A_1(r)+B_1(r),$$

that is

$$F(u_n(r)+v_n(r))-F(\beta+\gamma)\le A_1(r)+B_1(r), \forall r\ge 0. \tag{3.3}$$

It follows from F^{-1} is increasing on $[0,\infty)$ and (3.3) that

$$u_n(r)+v_n(r)\le F^{-1}(F(\beta+\gamma)+A_1(r)+B_1(r)), \forall r\ge 0. \tag{3.4}$$

It follows from $F(\infty)=\infty$ that $F^{-1}(\infty)=\infty$. By (3.4), the sequences $\{u_n\}$ and $\{v_n\}$ are bounded and increasing on $[0,c_0]$ for arbitrary $c_0>0$. Thus, $\{u_n\}$ and $\{v_n\}$ have subsequences converging uniformly to u and v on $[0,c_0]$. By the arbitrariness of $c_0>0$, we see that (u,v) is a positive solution of (3.1), that is, (U,V) is an entire positive solution of (1.3). Notice $U(0)=\beta, V(0)=\gamma$ and $(\beta,\gamma)\in(0,\infty)\times(0,\infty)$ was chosen arbitrarily, it follows that (1.3) has infinitely many positive entire solutions.

(i) If $A_1(\infty)<\infty$ and $B_1(\infty)<\infty$, then

$$u(r)+v(r)\le F^{-1}(F(\beta+\gamma)+A_1(\infty)+B_1(\infty))<\infty,$$

which imply that U,V are the positive entire bounded solutions of (1.3).

(ii) If $A_2(\infty)=\infty=B_2(\infty)$, since

$$u(r)\ge\beta+f^{1/(p-1)}(\beta,\gamma)A_2(r), \quad v(r)\ge\gamma+g^{1/(q-1)}(\beta,\gamma)B_2(r), \quad \forall r\ge 0.$$

Thus we have

$$\lim_{r\to\infty}u(r)=\lim_{r\to\infty}v(r)=\infty$$

which yield U,V are the positive entire large solutions of (1.3). The proof of theorem is now completed.

Proof of Theorem 3. If condition (1.6) holds, then we have

$$F(u_n(r)+v_n(r))\le F(\beta+\gamma)+A_1(r)+B_1(r)\le F(\beta+\gamma)+A_1(\infty)+B_1(\infty)\le F(\infty)<\infty.$$

Since F^{-1} is strictly increasing on $[0,\infty)$, we have

$$u_n(r)+v_n(r)\le F^{-1}(F(\beta+\gamma)+A_1(\infty)+B_1(\infty))<\infty.$$

The last part of the proof is clear from the proof of Theorem 2. The proof of Theorem 3 is now finished.

4 Proofs of Theorem 4

(i) It follows from the proof of Theorem 3, we have

$$u_n(r)\le u_{n+1}(r)\le f^{1/(p-1)}(u_n(r),v_n(r))A_1(r)\le f^{1/(p-1)}(u_n(r)+v_n(r),u_n(r)+v_n(r))A_1(r), \tag{4.1}$$

and

$$v_n(r) \le v_{n+1}(r) \le g^{1/(q-1)}(u_n(r), v_n(r))B_1(r) \le g^{1/(q-1)}(u_n(r)+v_n(r), u_n(r)+v_n(r))B_1(r). \quad (4.2)$$

Let $R > 0$ be arbitrary. From (4.1) and (4.2) we get

$$\begin{aligned} u_n(R) + v_n(R) &\le \beta + \gamma + f^{1/(p-1)}(u_n(R) + v_n(R), u_n(R) + v_n(R))A_1(R) \\ &\quad + g^{1/(q-1)}(u_n(R) + v_n(R), u_n(R) + v_n(R))B_1(R) \\ &\le \beta + \gamma + [f(u_n(R) + v_n(R), u_n(R) + v_n(R)) \\ &\quad + g(u_n(R) + v_n(R), u_n(R) + v_n(R))]^{1/(m_0-1)}(A_1(R) + B_1(R)), \quad n \ge 1. \end{aligned}$$

This implies

$$\begin{aligned} 1 &\le \frac{\beta+\gamma}{u_n(R) + v_n(R)} \\ &\quad + \frac{[f(u_n(R) + v_n(R), u_n(R) + v_n(R)) + g(u_n(R) + v_n(R), u_n(R) + v_n(R))]^{1/(m_0-1)}}{u_n(R) + v_n(R)} \\ &\quad \times (A_1(R) + B_1(R)), \quad n \ge 1. \end{aligned}$$

Taking into account the monotonicity of $(u_n(R) + v_n(R))_{n\ge1}$, there exists

$$L(R) := \lim_{n\to\infty}(u_n(R) + v_n(R)).$$

We claim that $L(R)$ is finite. Indeed, if not, we let $n \to \infty$ and the assumption (1.7) leads us to a contradiction. Thus $L(R)$ is finite. since u_n, v_n are increasing functions, it follows that the map $L : (0,\infty) \to (0,\infty)$ is nondecreasing and

$$u_n(r) + v_n(r) \le u_n(R) + v_n(R) \le L(R), \quad \forall r \in [0, R], \quad n \ge 1.$$

Thus the sequences $(u_n)_{n\ge1}, (v_n)_{n\ge1}$ are bounded from above on bounded sets. Let

$$u(r) := \lim_{n\to\infty} u_n(r), \quad v(r) := \lim_{n\to\infty} v_n(r), \quad for \quad r \ge 0.$$

Then (u, v) is a positive solution of (3.1).

In order to conclude the proof, it is enough to show that (u, v) is a large solution of (3.1). We see

$$u(r) \ge \beta + f^{1/(p-1)}(\beta,\gamma)A_2(r), \quad v(r) \ge \gamma + g^{1/(q-1)}(\beta,\gamma)B_2(r), \quad \forall r \ge 0.$$

Since f and g are positive functions and

$$A_2(\infty) = \infty = B_2(\infty) = \infty,$$

we can conclude that (u, v) is a large solution of (3.1) and so (U, V) is a positive entire large solution of (1.3). Thus any large solution of (3.1) provide a positive entire large solution (U, V) of (1.3) with $U(0) = \beta, V(0) = \gamma$. Since $(\beta,\gamma) \in (0,\infty) \times (0,\infty)$ was chosen arbitrarily, it follows that (1.3) has infinitely many positive entire large solutions.

(ii)If

$$\sup_{s\ge0}(f(s, s) + g(s, s))^{1/(m_0-1)} < \infty$$

holds, then we have

$$L(R) := \lim_{n\to\infty}(u_n(R) + v_n(R)) < \infty.$$

Thus

$$u_n(r) + v_n(r) \leq u_n(R) + v_n(R) \leq L(R), \quad \forall r \in [0, R], \quad n \geq 1.$$

So the sequences $(u_n)_{n\geq 1}, (v_n)_{n\geq 1}$ are bounded from above on bounded sets. Let

$$u(r) := \lim_{n\to\infty} u_n(r), \quad v(r) := \lim_{n\to\infty} v_n(r), \quad \text{for} \quad r \geq 0.$$

Then (u, v) is a positive solution of (3.1).

It follows from (4.1) and (4.2) that (u, v) is bounded, which implies that (1.3) has infinitely many positive entire bounded solutions. The proof is end.

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Competing Interests

The authors declare that no competing interests exist.

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