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Modified Solutions of Linear Differential Equations with Polynomial Coefficients near the Origin and Infinity

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

Linear differential equations with polynomial coefficients are studied. Solutions near the origin and infinity are presented for the differential equations of the second order and with two blocks of classified terms, where the solutions $u(t)$ near the origin and infinity are assumed to be expressed by a power series of t and t^{-1}, respectively, multiplied by a power of t. In the present study, it is shown that the function which is obtained from any of these solutions by multiplying $e^{\beta t}$ or $e^{\beta/t}$ or $(1-t/\alpha)^{\beta}$, is a solution of a differential equation with two or three blocks of classified terms, where α and β are constants. Discussions are given also of multipliers $e^{\beta t^2}$ or e^{β/t^2}. The studies are mainly made for the cases in which the singularities of the differential equation do not change, but some studies are given for the cases when the singularities change.

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1 Introduction

In [1, 2], differential equations of order $l_x \in \mathbb{Z}_{>0}$, with coefficients of polynomials, are studied. They take the form:

$$\sum_{k=0}^{l_x}\sum_{m=0}^{\infty} a_{k,m}t^m \frac{d^k}{dt^k}u(t) = \sum_{k=0}^{l_x}(a_{k,0} + a_{k,1}\cdot t + a_{k,2}\cdot t^2 + a_{k,3}\cdot t^3 + \cdots)\cdot \frac{d^k}{dt^k}u(t)$$
$$= 0, \quad t > 0, \tag{1.1}$$

where $a_{k,m}$ for $k \in \mathbb{Z}_{[0,l_x]}$ and $m \in \mathbb{Z}_{>-1}$ are constants. It was assumed that a finite number of the constants are nonzero.

Here $\mathbb{R}$ and $\mathbb{Z}$ are the sets of all real numbers and all integers, respectively, and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} | a \le n \le b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{C}$ which is the set of all complex numbers, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} | n > a\}$, $\mathbb{Z}_{<a} = \{n \in \mathbb{Z} | n < a\}$ for $a \in \mathbb{Z}$, and $\mathbb{R}_{>a} = \{x \in \mathbb{R} | x > a\}$ for $a \in \mathbb{R}$.

In [1], the terms of Equation (1.1) are reassembled as

$$\sum_{l=-\infty}^{l_x} \tilde{D}_t^l u(t) = 0, \quad t > 0, \tag{1.2}$$

where

$$\tilde{D}_t^l u(t) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l}\cdot t^{k-l}\frac{d^k}{dt^k}u(t), \tag{1.3}$$

each of $\tilde{D}_t^l u(t)$ is called a block of classified terms.

When $l_x = 2$, Equation (1.2) is expressed as

$$\tilde{D}_t^2 u(t) + \tilde{D}_t^1 u(t) + \tilde{D}_t^0 u(t) + \tilde{D}_t^{-1} u(t) + \tilde{D}_t^{-2} u(t) + \cdots = 0, \quad t > 0, \tag{1.4}$$

where

$$\tilde{D}_t^2 = a_{2,0}\cdot\frac{d^2}{dt^2}, \quad \tilde{D}_t^1 = a_{2,1}\cdot t\frac{d^2}{dt^2} + a_{1,0}\cdot\frac{d}{dt}, \quad \tilde{D}_t^0 = a_{2,2}\cdot t^2\frac{d^2}{dt^2} + a_{1,1}\cdot t\frac{d}{dt} + a_{0,0},$$
$$\tilde{D}_t^{-1} = a_{2,3}\cdot t^3\frac{d^2}{dt^2} + a_{1,2}\cdot t^2\frac{d}{dt} + a_{0,1}t, \quad \tilde{D}_t^{-2} = a_{2,4}\cdot t^4\frac{d^2}{dt^2} + a_{1,3}\cdot t^3\frac{d}{dt} + a_{0,2}t^2, \quad \cdots. \tag{1.5}$$

When we discuss a differential equation of order l_x, the following condition is adopted.

Condition 1.1. We consider such a differential equation is not regarded as a differential equation of $u'(t)$, so that $\sum_{m=0}^{\infty}|a_{l_x,m}| \ne 0$ and $\sum_{m=0}^{\infty}|a_{0,m}| \ne 0$.

In [1], special attention is focussed on Equation (1.4) for the case in which there exist two nonzero blocks of classified terms, so that the equation is expressed as

$$\tilde{D}_t^l u(t) + \tilde{D}_t^{l-m} u(t) = 0, \quad m \in \mathbb{Z}_{>0}, \tag{1.6}$$

Remark 1.1. By Equation (1.5) for $l_x = 2$, we see that Equation (1.6) for $l = -1, -2, \dots$ are equivalent to the one for $l = 0$, and the differential equation for $l = 1$ is equivalent to the one for $l = 0$ when $a_{0,0} = 0$. We note that the differential equation for $l = 2$ is equivalent to a special one for $l = 0$. Hence we study only the differential equation for $l = 0$.

In [1, 2], special attention is focussed on the solutions of

$$\tilde{D}_t^0 u(t) + \tilde{D}_t^{-1} u(t) = 0, \tag{1.7}$$

for $l_x = 2$.

We study the differential equations belonging to Equation (1.7) for $l = 0$ which are given by

$$\begin{aligned} {}_2D_t^0 u(t) + {}_3D_t^{-1} u(t) &= \delta_2' \cdot {}_2D_t^0(a,b)u(t) + \delta_1' \cdot {}_1D_t^0(a)u(t) - \delta_2 t \cdot {}_2D_t^0(\tilde{a},\tilde{b})u(t) - \delta_1 t \cdot {}_1D_t^0(\tilde{a})u(t) \\ &= [\delta_2' \cdot {}_2D_t^0(a,B) - \delta_2 t \cdot {}_2D_t^0(\tilde{a},\tilde{B})]u(t) = 0, \end{aligned} \tag{1.8}$$

$${}_2D_t^0 u(t) + {}_2D_t^{-1} u(t) = [{}_2\mathcal{D}_t^0(a,b) - \delta_1 t \cdot {}_1\mathcal{D}_t^0(\bar{c}) - \delta_0 t] = [{}_2\mathcal{D}_t^0(a,b) - \delta_1 t \cdot {}_1\mathcal{D}_t^0(\tilde{C})]u(t) = 0, \tag{1.9}$$

$${}_1D_t^0 u(t) + {}_3D_t^{-1} u(t) = [\delta_1 \cdot {}_1\mathcal{D}_t^0(c) - \delta_0 + t \cdot {}_2\mathcal{D}_t^0(\tilde{a},\tilde{b})]u(t) = [\delta_1 \cdot {}_1\mathcal{D}_t^0(C) + t \cdot {}_2\mathcal{D}_t^0(\tilde{a},\tilde{b})]u(t) = 0, \tag{1.10}$$

where δ_2', δ_1', δ_2, δ_1, δ_0, a, b, c, $B = b + \frac{\delta_0'}{\delta_1'}$, $C = c - \frac{\delta_0}{\delta_1}$, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{B} = \tilde{b} + \frac{\delta_0}{\delta_1}$ and $\tilde{C} = \tilde{c} + \frac{\delta_0}{\delta_1}$ are constants, and

$${}_2\mathcal{D}_t^0(a,b) = t^2 \frac{d^2}{dt^2} + (1+a+b)t\frac{d}{dt} + ab, \quad {}_1\mathcal{D}_t^0(c) = t \cdot \frac{d}{dt} + c. \tag{1.11}$$

In [2], we studied the solutions of Equations (1.8)∼(1.10), by using the equations which are obtained from them by putting $x = \frac{1}{t}$ and $\tilde{u}(x) = u(t)$. They are obtained with the aid of the following lemma.

Lemma 1.1. *Let $x = \frac{1}{t}$, $\tilde{u}(x) = u(t)$, and ${}_1\mathcal{D}_t^0(c)$ and ${}_2\mathcal{D}_t^0(a,b)$ be given in Equation* (1.11)*. Then*

$$t\frac{d}{dt}u(t) = -x\frac{d}{dx}\tilde{u}(x), \quad t^2\frac{d^2}{dt^2}u(t) = [x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx}]\tilde{u}(x), \tag{1.12}$$

$${}_1\mathcal{D}_t^0(c)u(t) = -{}_1\mathcal{D}_x^0(-c)\tilde{u}(x), \quad {}_2\mathcal{D}_t^0(a,b)u(t) = {}_2\mathcal{D}_x^0(-a,-b)\tilde{u}(x). \tag{1.13}$$

As a consequence, corresponding to Equations (1.8)∼(1.10), we have the following equations:

$$\begin{aligned} &[-\delta_2 \cdot {}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{b}) + \delta_1 \cdot {}_1D_t^0(-\tilde{a}) + \delta_2' x \cdot {}_2\mathcal{D}_x^0(-a,-b) - \delta_1' x \cdot {}_1D_t^0(-a)]\tilde{u}(x) \\ &\qquad = [-\delta_2 \cdot {}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{B}) + \delta_2' x \cdot {}_2\mathcal{D}_x^0(-a,-B)]\tilde{u}(x) = 0, \end{aligned} \tag{1.14}$$

$$[\delta_1 \cdot {}_1\mathcal{D}_x^0(-\bar{c}) - \delta_0 + x \cdot {}_2\mathcal{D}_x^0(-a,-b)]\tilde{u}(x) = [\delta_1 \cdot {}_1\mathcal{D}_x^0(-\tilde{C}) + x \cdot {}_2\mathcal{D}_x^0(-a,-b)]\tilde{u}(x) = 0, \tag{1.15}$$

$$[{}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{b}) - \delta_1 x \cdot {}_1\mathcal{D}_x^0(-c) - \delta_0 x]\tilde{u}(x) = [{}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{b}) - \delta_1 x \cdot {}_1\mathcal{D}_x^0(-C)]\tilde{u}(x) = 0. \tag{1.16}$$

Remark 1.2. We note that (i) Equation (1.16) is obtained from Equation (1.9), by replacing t by x, $u(t)$ by $\tilde{u}(x)$, a by $-\tilde{a}$, b by $-\tilde{b}$, $\tilde{c}$ by $-c$, and $\tilde{C}$ by $-C$, (ii) Equation (1.15) is obtained from Equation (1.10), by replacing t by x, $u(t)$ by $\tilde{u}(x)$, $\tilde{a}$ by $-a$, $\tilde{b}$ by $-b$, c by $-\tilde{c}$, and C by $-\tilde{C}$, and (iii) Equation (1.14) is obtained from Equation (1.8), by replacing t by x, $u(t)$ by $\tilde{u}(x)$, a by $-\tilde{a}$, b by $-\tilde{b}$, $\tilde{a}$ by $-a$, $\tilde{b}$ by $-b$, and δ_2 by $\frac{1}{\delta_2}$.

Notation 1.1. *We denote Equation* (1.9) *for $\delta_1 \neq 0$ and $\delta_0 = 0$, and for $\delta_1 = 0$ and $\delta_0 \neq 0$, by Equations* (1.9-1) *and* (1.9-0)*, respectively, and those for Equation* (1.10) *by Equations* (1.10-1) *and* (1.10-0)*, those for Equation* (1.15) *by Equations* (1.15-1) *and* (1.15-0)*, and those for Equation* (1.16) *by Equations* (1.16-1)*. and* (1.16-0)*.*

The special one of Equation (1.8) given by

$$(t\frac{d^2}{dt^2}+c\frac{d}{dt})u(t)-[t^2\frac{d^2}{dt^2}+(1+\tilde{a}+\tilde{b})t\frac{d}{dt}+\tilde{a}\tilde{b}]u(t)=0, \tag{1.17}$$

is the hypergeometric differential equation. The hypergeometric function given by

$${}_2F_1(\tilde{a},\tilde{b};c;t)=\sum_{k=0}^{\infty}\frac{(\tilde{a})_k(\tilde{b})_k}{k!(c)_k}t^k, \tag{1.18}$$

is a solution of Equation (1.17), where $(a)_k$ for $a\in\mathbb{C}$ and $k\in\mathbb{Z}_{>-1}$ denote $(a)_k=\prod_{m=0}^{k-1}(a+m)$ if $k>0$, and $(a)_0=1$ if $k=0$.

The special one of Equation (1.9-1) given by

$$(t\frac{d^2}{dt^2}+c\frac{d}{dt})u(t)-(t\frac{d}{dt}+\tilde{a})u(t)=0, \tag{1.19}$$

is Kummer's differential equation. The confluent hypergeometric function given by

$${}_1F_1(\tilde{a};c;t)=\sum_{k=0}^{\infty}\frac{(\tilde{a})_k}{k!(c)_k}t^k, \tag{1.20}$$

is a solution of Equation (1.19).

The special one of Equation (1.9-0) given by

$$(t\frac{d^2}{dt^2}+c\frac{d}{dt})u(t)-u(t)=0, \tag{1.21}$$

has the solution given by

$${}_0F_1(;c;z)=\sum_{k=0}^{\infty}\frac{1}{k!(c)_k}t^k. \tag{1.22}$$

The special one of Equation (1.10-1) given by

$$\frac{d}{dt}u(t)-[t^2\frac{d^2}{dt^2}+(1+\tilde{a}+\tilde{b})t\frac{d}{dt}+\tilde{a}\tilde{b}]u(t)=0, \tag{1.23}$$

has the solution given by

$${}_2F_0(\tilde{a},\tilde{b};;t)=\sum_{k=0}^{\infty}\frac{(\tilde{a})_k(\tilde{b})_k}{k!}t^k. \tag{1.24}$$

Remark 1.3. The functions in Equations (1.18), (1.20) and (1.22), are divergent at $t\neq 0$, if $c\in\mathbb{Z}_{<1}$. The function in Equation (1.20) is a polynomial if $a\in\mathbb{Z}_{<1}$, and the functions in Equations (1.18) and (1.24), are polynomials if $a\in\mathbb{Z}_{<1}$ or $b\in\mathbb{Z}_{<1}$. If otherwise, they are infinite series. Then as functions of $t\in\mathbb{C}$, the functions in Equations (1.20) and (1.22), are entire functions, and the functions in Equations (1.18) and (1.24), have radius 1 and 0, respectively, of convergence.

Remark 1.4. Laguerre's differential equation is a special one of Kummer's differential equation; see Chapter VIII in [3], and Chapter 13 in [4].

In [5, 6, 7], the solutions of Kummer's and the hypergeometric differential equation, given by Equations (1.19) and (1.17), were studied with the aid of distribution theory, and of the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation. In the study, the following condition was adopted.

Condition 1.2. $u(t)$ is expressed as a linear combination of $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ for $t > 0$ and $\nu \in S$, where S is a set of $\nu \in \mathbb{R}_{>-M} \backslash \mathbb{Z}_{<1}$ for some $M \in \mathbb{Z}_{>-1}$.

We then express $u(t)$ as follows:

$$u(t) = \sum_{\nu \in S} u_{\nu-1} \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{1.25}$$

where $u_{\nu-1} \in \mathbb{C}$ are constants. Because of this condition, we obtained the solutions which are expressed by a power series of t multiplied by a power t^α:

$$u(t) = t^\alpha \sum_{k=0}^{\infty} p_k t^k, \tag{1.26}$$

where $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{<0}$, $p_k \in \mathbb{C}$ and $p_0 \neq 0$. The solutions are obtained in [1], by the method of Frobenius; see Section 2.4.1 in [8].

In [1], every equation of Equation (1.6) for $m \in \mathbb{Z}_{>1}$ is shown to reduce to a differential equation of the form of Equation (1.7) by a change of variable. The following discussions are related with Equations (1.8)$\sim$(1.10), except in Section 3.4 where a case of $m = 2$ is treated.

In [9], the asymptotic behaviors as $t \to \infty$ are discussed for the confluent hypergeometric function, which is a solution of Kummer's differential equation, in the standpoint of fractional calculus.

In [2], solutions $u(t)$ of Equations (1.8)$\sim$(1.10), near infinity, are obtained with the aid of corresponding solutions $\tilde{u}(x)$ of Equations (1.14)$\sim$(1.16), around the origin, in the form:

$$u(t) = t^\alpha \sum_{k=0}^{\infty} q_k t^{-k} = \tilde{u}(x) = x^{-\alpha} \sum_{k=0}^{\infty} q_k x^k, \quad t = \frac{1}{x} \to \infty, \tag{1.27}$$

where $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{<0}$, $q_k \in \mathbb{C}$ and $q_0 \neq 0$ are constants. Then the solutions $\tilde{u}(x)$ of the latter, near infinity, are obtained from the solutions $u(t)$ of the former, around the origin, in the form:

$$\tilde{u}(x) = x^{-\alpha} \sum_{k=0}^{\infty} p_k x^{-k} = u(t) = t^\alpha \sum_{k=0}^{\infty} p_k t^k, \quad x = \frac{1}{t} \to \infty. \tag{1.28}$$

1.1 Singular points of Equations (1.8)$\sim$(1.10) and (1.14)$\sim$(1.16)

Remark 1.5. In Section 2.4 of [8] and Section 7.21 of [10], terminologies "regular singular point" and "irregular singular point" are used. For a differential equation, which is expressed by

$$[(t-c)^2 \frac{d^2}{dt^2} + (t-c)p(t)\frac{d}{dt} + q(t)]u(t) = 0, \tag{1.29}$$

where c is a constant, and $p(t)$ and $q(t)$ are rational functions of t. The point $t = c$ is called a singular point, if $\frac{p(t)}{t-c}$ or $\frac{q(t)}{(t-c)^2}$ is not analytic at c. If the point $t = c$ is a singular point, it is said to be regular or irregular at $t = c$, according as both $p(t)$ and $q(t)$ are analytic in a neighborhood of the point $t = c$, or not so.

In discussing the solutions around a singular point c, s-rank $R(c)$ was introduced by

$$R(c) = \max\{1, K_1(c), K_2(c)/2\}, \tag{1.30}$$

in [11], and by $R(c) = \max\{K_1(c), K_2(c)/2\}$ in [12], where $K_1(c)$ and $K_2(c)$, respectively, are the multiplicities of the poles of $\frac{p(t)}{t-c}$ and $\frac{q(t)}{(t-c)^2}$ at $t = c$. According as the singular point is regular or irregular, $R(c) \leq 1$ or $R(c) > 1$.

Notation 1.2. *When $c \neq \infty$, we use notations $R(c)$, $K_1(c)$ and $K_2(c)$ for Equations (1.8)∼(1.10), and notations $\tilde{R}(c)$, $\tilde{K}_1(c)$ and $\tilde{K}_2(c)$, in place of $R(c)$, $K_1(c)$ and $K_2(c)$, for Equations (1.14)∼(1.16). When $c = \infty$, we put $R(\infty) = \tilde{R}(0)$, $K_1(\infty) = \tilde{K}_1(0)$, $K_2(\infty) = \tilde{K}_2(0)$, $\tilde{R}(\infty) = R(0)$, $\tilde{K}_1(\infty) = K_1(0)$ and $\tilde{K}_2(\infty) = K_2(0)$, for each pair of Equations (1.8) and (1.14), (1.9) and (1.15), and (1.10) and (1.16).*

Lemma 1.2. *We present a method of obtaining $K_1(0)$ and $K_2(0)$ for a differential equation of the form:*

$$[P_0(t)t^2 \cdot \frac{d^2}{dt^2} + P_1(t)t \cdot \frac{d}{dt} + P_2(t)]u(t) = 0, \tag{1.31}$$

where $P_0(t)$, $P_1(t)$ and $P_2(t)$ are polynomials of t. Let p_0 be the power of t in the lowest order term in $P_0(t)$, and p_1 and p_2 are those in $P_1(t)$ and $P_2(t)$, respectively. Then Remark 1.5 shows that $K_1(0)$ and $K_2(0)$ of this equation are given by $K_1(0) = p_0 + 2 - p_1 - 1$ and $K_2(0) = p_0 + 2 - p_2$.

Lemma 1.3. *We put $x = \frac{1}{t}$ and $\tilde{u}(x) = u(t)$. When $u(t)$ satisfies (1.31), by Lemma 1.1, the equation which $\tilde{u}(x)$ satisfies, is given by*

$$[P_0(\frac{1}{x})x^2 \cdot \frac{d^2}{dx^2} + 2P_0(\frac{1}{x})x \cdot \frac{d}{dx} - P_1(\frac{1}{x})x \cdot \frac{d}{dx} + P_2(\frac{1}{x})]\tilde{u}(x) = 0. \tag{1.32}$$

Let q_0 be the power of t in the highest order term in $P_0(t)$, and q_1 and q_2 are those in $P_1(t)$ and $P_2(t)$, respectively. Then $\tilde{K}_1(0)$ and $\tilde{K}_2(0)$ of this equation are given by $\tilde{K}_1(0) = \text{Max}\{q_1 - q_0 + 1, 1, 0\}$ and $\tilde{K}_2(0) = q_2 - q_0 + 2$.

Remark 1.6. By the terminology given in Remark 1.5, the point $t = 0$ is a regular singular point of Equations (1.8) and (1.9), and it is an irregular singular point of Equations (1.10-1) and (1.10-0). In fact, for the latter two equations, we have $R(0) = 2$, $K_1(0) = 2$ and $K_2(0) = 3$ or 2, and $R(0) = \frac{3}{2}$, $K_1(0) = 1$ and $K_2(0) = 3$, respectively. Applying Frobenius' method, we see that there exist two and one solutions of the form of Equation (1.26) for Equations (1.8) and (1.9), and Equation (1.10-1), respectively, and there exists no solution of that form for Equation (1.10-0).

Remark 1.7. The point $x = 0$ is a regular singular point of Equations (1.14) and (1.16), and an irregular singular point of Equations (1.15-1) and (1.15-0). In fact, for the latter two equations, we have $\tilde{R}(0) = 2$, $\tilde{K}_1(0) = 2$ and $\tilde{K}_2(0) = 3$ or 2, and $\tilde{R}(0) = \frac{3}{2}$, $\tilde{K}_1(0) = 1$ and $\tilde{K}_2(0) = 3$, respectively. Applying Frobenius' method, we see that there exist two and one solutions of the form of Equation (1.27) for $\tilde{u}(x)$, for Equations (1.14) and (1.16), and Equation (1.15-1), respectively, and there exists no solution of that form for Equation (1.15-0).

From Notation 1.2 and these remarks, we obtain the following remarks.

Remark 1.8. The point $x = \infty$ is a regular singular point of Equations (1.14) and (1.15), and it is an irregular singular point of Equation (1.16). There exist two and one solutions of the form of Equation (1.28) for $\tilde{u}(x)$ for Equations (1.14) and (1.15), and Equation (1.16-1), respectively, and there exists no solution of that form for Equation (1.16-0).

Remark 1.9. The point of $t = \infty$ is a regular singular point of Equations (1.8) and (1.10), and an irregular singular point of Equation (1.9). There exist two and one solutions of the form of Equation (1.27) for $u(t)$, for Equations (1.8) and (1.10), and Equation (1.9-1), respectively, and there exists no solution of that form for Equation (1.9-0).

In Section 2, we present a theorem which summarizes the results for the solution of the forms of Equations (1.26) and (1.27) for Equations (1.8)∼(1.10), and those of the forms of Equations (1.27) and (1.28) for Equations (1.14)∼(1.16), given in [1, 2].

1.2 Transformation of a differential equation via a function

Remark 1.10. In Section 7.3 of [10], discussion is made of the asymptotic solution for the case when the point of infinity is an irregular singular point.

For Equation (1.9), $R(\infty) = K_1(\infty) = K_1(\infty) = 2$ and then we obtain the asymptotic solution of the form:

$$u(t) = e^{\beta_1 t} t^{\alpha} \sum_{k=0}^{\infty} q_k t^{-k}, \tag{1.33}$$

where α and β_1 are constants.

For Equation (1.10), the origin is an irregular singular point. In this case, $R(0) = K_1(0) = K_1(0) = 2$ we have a solution of the form:

$$u(t) = e^{\beta_1/t} t^{\alpha} \sum_{k=0}^{\infty} p_k t^{k}, \tag{1.34}$$

by the corresponding argument.

In discussing the solutions of Kummer's differential equation, we often meet solutions of the form

$$e^{\beta_1 t} t^{\alpha} \sum_{k=0}^{\infty} p_k t^{k}, \tag{1.35}$$

where α and β_1 are constants. We also meet such differential equations, that the equations are not of the form of Equations (1.8)∼(1.10), but their solutions are of the form of Equation (1.35). This type of problems are discussed in Section 1.1.2 of [12].

Remark 1.11. In [12], discussion is presented on the equations of the form:

$$L_z y(z) := [P_0(z)D_z^2 + P_1(z)D_z + P_2(z)]y(z) = 0, \tag{1.36}$$

where $P_0(z)$, $P_1(z)$ and $P_2(z)$ are polynomials of z, and D_z represents $\frac{d}{dz}$. For these equations, s-homotopic transformation, that is a transformation without increasing the singularities, is proposed, where function $G(z)$ is introduced, and then transformations of L_z and $y(z)$, $\tilde{L}_z$ and $v(z)$, are defined by $\tilde{L}_z = G(z)L_z G(z)^{-1}$ and $v(z) = G(z)y(z)$, so that Equation (1.36) is transformed to $\tilde{L}_z v(z) = 0$. In [12], a general discussion is given on this problem, where Kummer's equation and the hypergeometric equation are treated as special examples. We now call this transformation of a differential equation the transformation via multiplier or function $G(z)$.

In Section 3, we present the transformation of Equation (1.9) via multiplier $e^{\beta_1 t}$ or $e^{\beta_2 t^2}$, where β_1 and β_2 are constants. In Section 4, we present the corresponding study for Equation (1.10). As a consequence, we obtain a set of solutions of the form of Equation (1.26) or (1.27), and those multiplied by $e^{\beta_1 t}$ or $e^{\beta_1/t}$, for Equations (1.9) and (1.10). In these cases, the original equations have one regular singularity at 0 and one irregular one at ∞, which are not changed by the transformations.

In discussing the solution of the hypergeometric differential equation, we often write solutions which are of the form of Equation (1.26) multiplied by $(1-t)^{\beta}$.

In Section 5, we obtain a transformed equation of Equation (1.8), by a transformation via multiplier $(1-t)^{\beta}$. As a consequence, we obtain a set of solutions of the form of Equation (1.26) or (1.27), and those multiplied by $(1-t)^{\beta}$, for Equation (1.8). In these cases, the original equations have two regular singularities at 0 and 1, and one irregular one at ∞, which are not changed by the transformations.

In Section 3.6, we present the transformation of Equations (1.9) via $(1-t)^{\beta}$. In this case, the original equation has one regular singularity at 0 and one irregular one at ∞, but the transformed equation has two regular singularities at 0 and 1, and one irregular one at ∞. We note that the transformed equation for $\beta = -1$ is an example of the differential differential equation for which Stewart [14] discussed the asymptotic form of the solution around ∞. In the present example, we have explicit expressions of the solution.

In Sections 5.3 and 5.4, we present the transformation of Equation (1.8) via $e^{\beta_1 t}$ and $e^{\beta_1/t}$, respectively. In these cases, the original equation has three regular singularites at 0, 1 and ∞, but the transformed equation has two regular singularities at 0 and 1, and one irregular one at ∞.

2 Solutions of Equations (1.8)∼(1.10) and (1.14)∼(1.16)

The following theorem is based on Theorem 2.2 in [1], and Theorem 2.1 and Sections 3.1∼3.5 in [2].

Theorem 2.1. *We have the following solutions of the forms of Equations* (1.26) *and* (1.27) *for Equations* (1.8)∼(1.10) *and* (1.14)∼(1.16), *where* $x = \frac{1}{t}$.

(i). *If* $a - b \notin \mathbb{Z}$ *and* $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, *we have two pairs of solutions of Equation* (1.8):

$$\phi_{-\alpha}(\frac{\delta_2}{\delta_2'}t) := (\frac{\delta_2}{\delta_2'}t)^{-\alpha} \cdot {}_2F_1(\tilde{a}-\alpha, \tilde{B}-\alpha; 1+a+B-2\alpha; \frac{\delta_2}{\delta_2'}t), \quad \alpha = a,\ B, \tag{2.1}$$

$$\tilde{\phi}_{\tilde{\alpha}}(\frac{\delta_2'}{\delta_2}x) := (\frac{\delta_2'}{\delta_2}x)^{\tilde{\alpha}} \cdot {}_2F_1(\tilde{\alpha}-a, \tilde{\alpha}-B; 1-\tilde{a}-\tilde{B}+2\tilde{\alpha}; \frac{\delta_2'}{\delta_2}x), \quad \tilde{\alpha} = \tilde{a},\ \tilde{B}. \tag{2.2}$$

Here the solutions given in (2.2) *are those of Equation* (1.14).

(ii). *If* $a - b \notin \mathbb{Z}$ *and* $\delta_1 \neq 0$, *we have one pair and one solutions of Equation* (1.9):

$$\phi_{-\alpha}(\delta_1 t) := (\delta_1 t)^{-\alpha} \cdot {}_1F_1(\tilde{C}-\alpha; 1+a+b-2\alpha; \delta_1 t), \quad \alpha = a,\ b. \tag{2.3}$$

$$\tilde{\psi}_{2,\tilde{C}}(\frac{1}{\delta_1}x) := (\frac{1}{\delta_1}x)^{\tilde{C}} \cdot {}_2F_0(\tilde{C}-a, \tilde{C}-b; ; -\frac{1}{\delta_1}x), \tag{2.4}$$

where $\tilde{C} = \tilde{c} + \frac{\delta_0}{\delta_1}$. *Here the solution given by* (2.4) *is a solution of Equation* (1.15).

(iii). *If* $a - b \notin \mathbb{Z}$, *we have one pair of solutions of Equation* (1.9-0):

$$\phi_{-\alpha}(\delta_0 t) := (\delta_0 t)^{-\alpha} \cdot {}_0F_1(; 1+a+b-2\alpha; \delta_0 t), \quad \alpha = a,\ b. \tag{2.5}$$

(iv). *If* $\tilde{a} - \tilde{b} \notin \mathbb{Z}$ *and* $\delta_1 \neq 0$, *we have one and a pair of solutions of Equation* (1.10):

$$\psi_{2,-C}(\frac{1}{\delta_1}t) := (\frac{1}{\delta_1}t)^{-C} \cdot {}_2F_0(\tilde{a}-C, \tilde{b}-C; ; -\frac{1}{\delta_1}t), \tag{2.6}$$

$$\tilde{\phi}_{\tilde{\alpha}}(\delta_1 x) := (\delta_1 x)^{\tilde{\alpha}} \cdot {}_1F_1(\tilde{\alpha}-C; 1-\tilde{a}-\tilde{b}+2\tilde{\alpha}; \delta_1 x), \quad \tilde{\alpha} = \tilde{a},\ \tilde{b}, \tag{2.7}$$

where $C = c - \frac{\delta_0}{\delta_1}$. *Here the solutions given in* (2.7) *are those of Equation* (1.16).

(v). *If* $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, *there exist a pair of solutions of Equation* (1.10-0):

$$\tilde{\phi}_{\tilde{\alpha}}(\delta_0 x) := (\delta_0 x)^{\tilde{\alpha}} \cdot {}_0F_1(; 1-\tilde{a}-\tilde{b}+2\tilde{\alpha}; \delta_0 x), \quad \tilde{\alpha} = \tilde{a},\ \tilde{b}. \tag{2.8}$$

Here the solutions given in (2.8) *are those of Equation* (1.16-0).

A proof of this theorem is given in Section 2.1.

Remark 2.1. In Theorem 2.1 (i), (ii) and (iii), statement "$a - b \notin \mathbb{Z}$" appears. When $n := -a - (-b) \in \mathbb{Z}_{>-1}$, we always have solution $\phi_{-a}(t)$, but have solution $\phi_{-b}(t)$ only if the power series in it is a polynomial of degree less than n. This type of statement may appear every time when we have two solutions of the form of Equation (1.26) or (1.27), but it is often omitted in the following.

2.1 Proof of Theorem 2.1

2.1.1 Transformation of Equatiors (1.8)∼(1.10) via a power of t

Lemma 2.1. *Let* ${}_2\mathcal{D}_t^0(a, b)$ *and* ${}_1\mathcal{D}_t^0(\tilde{C})$ *be given in Equation* (1.11), $u(t) = t^{-\alpha}u_\alpha(t)$ *and* $\alpha \in \mathbb{C}$. *Then*

$${}_2\mathcal{D}_t^0(a, b)u(t) = t^{-\alpha} \cdot {}_2\mathcal{D}_t^0(a-\alpha, b-\alpha)u_\alpha(t), \quad {}_1\mathcal{D}_t^0(\tilde{C})u(t) = t^{-\alpha} \cdot {}_1\mathcal{D}_t^0(\tilde{C}-\alpha)u_\alpha(t). \tag{2.9}$$

In particular, when $\alpha = a$, *we have*

$${}_2\mathcal{D}_t^0(0, b-a)u_a(t) = [t^2\frac{d^2}{dt^2} + (1+b-a)t\frac{d}{dt}]u_a(t). \tag{2.10}$$

Proof. When $u(t) = t^{-\alpha}u_\alpha(t)$,

$$u'(t) = t^{-\alpha}u'_\alpha(t) - \alpha t^{-\alpha-1}u_\alpha(t), \tag{2.11}$$
$$u"(t) = t^{-\alpha}u_\alpha"(t) - 2\alpha t^{-\alpha-1}u'_\alpha(t) + \alpha(\alpha+1)t^{-\alpha-2}u_\alpha(t). \tag{2.12}$$

By using these on the lefthand sides of Equation (2.9), we obtain the righthand sides. □

Remark 2.2. Let $u(t)$ be a solution of Equation (1.9), and $u_a(t) = t^a u(t)$. By comparing an equation which is obtained by using Equations (2.9) and (2.10) in Equation (1.9), with Equation (1.19), we see that $u_a(t)$ is the confluent hypergeomric function which appears in Equation (2.3) for $\alpha = a$. By using b in place of a, we obtain the corresponding result for $u_b(t)$. When $\delta_1 = 0$ and $\delta_0 \neq 0$, we use Equation (1.9-0) in place of Equation (1.9), and then by using Equation (1.21) in place of Equation (1.19), we obtain Equation (2.5).

Remark 2.3. Let $u(t)$ be a solution of Equation (1.8), and $u_a(t) = t^a u(t)$. By comparing an equation which is obtained by using the first equation in Equation (2.9) for $a = \tilde{a}$ and $b = \tilde{b}$ and Equation (2.10) in Equation (1.8), with Equation (1.17), we see that $u_a(t)$ is the hypergeomric function which appears in Equation (2.1) for $\alpha = a$. By using b in place of a, we obtain the corresponding result for $u_b(t)$.

Remark 2.4. Let $u(t)$ be a solution of Equation (1.10), and $u_C(t) = t^C u(t)$. By comparing the equation which is obtained from Equation (1.10), by using $a = \tilde{a}$, $b = \tilde{b}$ and $\tilde{C} = C$ in the equations in Equation (2.9), with Equation (1.23), we see that $u_C(t)$ is the function which appears in Equation (2.6).

2.1.2 Transformation of Equations (1.14)∼(1.16) via a power of x

Lemma 2.2. *Let ${}_2\mathcal{D}_t^0(a,b)$ and ${}_1\mathcal{D}_t^0(\tilde{C})$ be given in Equation (1.11), $\tilde{u}(x) = x^\alpha \tilde{v}(x)$, and $\alpha \in \mathbb{C}$. Then*

$${}_2\mathcal{D}_x^0(-a,-b)\tilde{u}(x) = {}_2\mathcal{D}_x^0(\alpha-a,\alpha-b)\tilde{v}(x), \quad {}_1\mathcal{D}_x^0(-\tilde{C})\tilde{u}(x) = {}_1\mathcal{D}_x^0(\alpha-\tilde{C})\tilde{v}(x). \tag{2.13}$$

Proof. We obtain (2.13) from (2.9) by replacing t by x, u by $\tilde{u}$, v by $\tilde{v}$, a by $-a$, b by $-b$, $\tilde{C}$ by $-\tilde{C}$, and α by $-\alpha$. □

Remark 2.5. Remark 1.2 shows that we can obtain (i) Equation (1.16) from Equation (1.9), (ii) Equation (1.15) from Equation (1.10), and (iii) Equation (1.14) from Equation (1.8). By using these processes, we can obtain (i) Equation (2.7) from Equation (2.3), and Equation (2.8) from Equation (2.5), (ii) Equation (2.4) from Equation (2.6), and (iii) Equation (2.2) from Equation (2.1).

2.2 Alternative representation of the solution of Equation (1.15)

In this place, following Equation 13.5.2 of [4], we introduce function $U(a;b;t)$ as follows.

Definition 2.1. Let the solution of Equation (1.9) be given by Equation (2.3). Then

$$U(\tilde{C}-\alpha; A_0-2\alpha; \delta_1 t) = (\delta_1 t)^{-\tilde{C}+\alpha}{}_2F_0(\tilde{C}-\alpha; \tilde{C}-\alpha-A_0+2\alpha+1; -\frac{1}{\delta_1 t})$$
$$= (\delta_1 t)^{-\tilde{C}+\alpha}{}_2F_0(\tilde{C}-a; \tilde{C}-b; -\frac{1}{\delta_1 t}), \quad \alpha = a,\ b. \tag{2.14}$$

Notation 2.1. *Let $\phi_{-\alpha}(\delta_1 t)$ be given by Equation (2.3), and $\tilde{\phi}_{-\alpha}(\delta_1 t)$ be given by the equation which is obtained from Equation (2.3), by replacing ϕ by $\tilde{\phi}$, and ${}_1F_1$ by U.*

Lemma 2.3. *By Definition 2.1 and Notation 2.1, we can use*

$$\tilde{\phi}_{-a}(\delta_1 t) = \tilde{\phi}_{-b}(\delta_1 t) := \tilde{\psi}_{2,\tilde{C}}(\frac{x}{\delta_1}) := (\frac{x}{\delta_1})^{\tilde{C}} \cdot {}_2F_0(\tilde{C}-a, \tilde{C}-b;; -\frac{x}{\delta_1}), \tag{2.15}$$

in place of Equation (2.4).

2.2.1 Alternative representation of the solution of Equation (1.10)

Definition 2.2. Let the solution of Equation (1.16) be given by Equation (2.7). Then

$$U(\tilde{\alpha}-C;\tilde{A}_0+2\tilde{\alpha};\delta_1 x)=(\delta_1 x)^{-\tilde{\alpha}+C}\cdot {}_2F_0(\tilde{a}-C,\tilde{a}-C-\tilde{A}_0-2\tilde{\alpha}+1;;-\frac{1}{\delta_1 x})$$
$$=(\delta_1 x)^{-\tilde{\alpha}+C}\cdot {}_2F_0(\tilde{a}-C,\tilde{b}-C;;-\frac{1}{\delta_1 x}),\quad \tilde{\alpha}=\tilde{a},\ \tilde{b}. \tag{2.16}$$

Notation 2.2. *Let $\tilde{\phi}_{\tilde{\alpha}}(\delta_1 t)$ be given by Equation* (2.7)*, and $\phi_{\tilde{\alpha}}(\delta_1 t)$ be given by the equation which is obtained from Equation* (2.7)*, by replacing $\tilde{\phi}$ by ϕ, and ${}_1F_1$ by U.*

Lemma 2.4. *By Definition 2.2 and Notation 2.2, we can use*

$$\phi_{\tilde{a}}(\delta_1 x)=\phi_{\tilde{b}}(\delta_1 x):=\psi_{2,-C}(\frac{1}{\delta_1}t):=(\frac{1}{\delta_1}t)^{-C}\cdot {}_2F_0(\tilde{a}-C,\tilde{b}-C;;-\frac{1}{\delta_1}t), \tag{2.17}$$

in place of Equation (2.6)*.*

2.3 Confluence of the solutions of Equations (1.8), (1.9) and (1.10)

Remark 2.6. We put $\delta_2'=1$, $\delta_1'=0$, $\delta_1\neq 0$, $\delta_2\neq 0$, $B=b$, $\tilde{B}=\tilde{b}+\frac{\delta_1}{\delta_2}$ and $\tilde{a}=\tilde{c}$ in Equation (1.8), and tend δ_2 to 0, and then we obtain Equation (1.9-1). As a consequence, we can confirm that in the limit $\delta_2\to 0$, the solutions given in Theorem 2.1(i) converge to those in Theorem 2.1(ii). In [12], this process is called the confluence process, where $\delta_2=1$. We there replace t by $\frac{t}{\tilde{b}}$, and $\tilde{a}$ by $\tilde{c}$ in Equation (1.8), tend $\tilde{b}$ to ∞, and then we obtain Equation (1.9-1).

Remark 2.7. We put $\delta_2=1$, $\delta_1=0$, $\delta_1'\neq 0$, $\delta_2'\neq 0$, $\tilde{B}=\tilde{b}$, $B=b+\frac{\delta_1'}{\delta_2'}$ and $a=c$ in Equation (1.8), and tend δ_2' to 0, and then we obtain Equation (1.10-1) for $\delta_1=\delta_1'$ and $\delta_0=0$. As a consequence, we can confirm that in the limit $\delta_2'\to 0$, the solutions given in Theorem 2.1(i) converge to those in Theorem 2.1(iv). We do not have a limit of $\phi_{b+\delta_1'/\delta_2'}(\frac{x}{\delta_2'})$ in Equation (2.1).

Remark 2.8. We put $\delta_1\neq 0$ and $\delta_0\neq 0$ in Equation (1.9), tend δ_1 to 0, and then we obtain Equation (1.9-0). As a consequence, in the limit $\delta_1\to 0$, we obtain the solutions given in Theorem 2.1(iii) from those in Theorem 2.1(ii). We do not have a limit of Equation (2.4).

Remark 2.9. We put $\delta_1\neq 0$ and $\delta_0\neq 0$ in Equation (1.10), tend δ_1 to 0, and then obtain Equation (1.10-0). As a consequence, in the limit $\delta_1\to 0$, we obtain the solutions given in Theorem 2.1(v) from those in Theorem 2.1(iv). We do not have a limit of Equation (2.6).

3 Transformations of Equations (1.9) and (1.15)

Equations (1.9) and (1.15) are expressed as follows:

$$[{}_2\mathcal{D}_t^0(a,b)-\delta_1 t\cdot {}_1\mathcal{D}_t^0(\tilde{C})]u(t)=[t^2\frac{d^2}{dt^2}+A_0 t\frac{d}{dt}+ab-\delta_1 t(t\frac{d}{dt}+\tilde{C})]u(t)=0, \tag{3.1}$$

$$[\delta_1\cdot {}_1\mathcal{D}_x^0(-\tilde{C})+x\cdot {}_2\mathcal{D}_x^0(-a,-b)]\tilde{u}(x)=\delta_1(x\frac{d}{dx}-\tilde{C})+x\cdot {}_2\mathcal{D}_x^0(-a,-b)]\tilde{u}(x)=0, \tag{3.2}$$

where $A_0=1+a+b$ and $\tilde{C}=\tilde{c}+\frac{\delta_0}{\delta_1}$.

Remark 3.1. By Lemmas 1.2 and 1.3, we have $K_1(0)=1$, $K_2(0)=2$, $\tilde{K}_1(0)=2$ and $\tilde{K}_2(0)=3$ for Equation (3.1), and hence $R(0)=\tilde{R}(\infty)=1$ and $R(\infty)=\tilde{R}(0)=2$. From these, we conclude that Equation (3.1) has a regular and an irregular singular point at $t=0$ and $t=\infty$, respectively, as mentioned in Remarks 1.6 and 1.9.

Notation 3.1. *In the following, we use notations* (3.1-1) *and* (3.1-0), *as in Notation 1.1. Here Equation* (3.1-0) *is obtained from Equation* (3.1) *by replacing* $\delta_1\tilde{C}$ *by* δ_0, *and the other* δ_1 *by* 0.

Lemma 3.1. *By Theorem 2.1(ii) and Lemma 2.3, if* $a-b\notin\mathbb{Z}$ *and* $\delta_1\neq 0$, *Equations* (3.1) *and* (3.2) *have the solutions given by Equations* (2.3) *and* (2.15), *respectively.*

Lemma 3.2. *By Theorem 2.1(iii), if* $a-b\notin\mathbb{Z}$, $\delta_1=0$, *and* $\delta_0\neq 0$, *Equation* (3.1-0) *has the solutions given by Equation* (2.5).

3.1 Transformation of Equation (3.1) via an exponential function

Theorem 3.1. *Let* $u(t)$ *be a solution of Equation* (3.1), $\beta_1\in\mathbb{C}\backslash\{0\}$, *and*

$$A_1=2\beta_1+\delta_1,\quad B_1=\beta_1 A_0+\delta_1\tilde{C},\quad B_2=\beta_1(\beta_1+\delta_1), \tag{3.3}$$

which satisfy $A_1^2-4B_2=\delta_1^2$. *Then* $v(t):=e^{\beta_1 t}u(t)$ *is a solution of*

$$[{}_2D_t^0(a,b)-t(A_1 t\frac{d}{dt}+B_1)+B_2t^2]v(t)=0. \tag{3.4}$$

In the following, we use notations (3.4-1) *and* (3.4-0), *as in Notation 3.1.*

A proof of this theorem is given in Section 3.7.

3.1.1 Transformation of Equation (3.1) via a special exponential function

When $\delta_1\neq 0$, we choose $\beta_1=-\delta_1$, and then $B_2=0$, and Theorem 3.1 becomes

Lemma 3.3. *Let* $u(t)$ *be a solution of Equation* (3.1). *Then* $v_1(t)=e^{-\delta_1 t}u(t)$ *is a solution of*

$$[{}_2D_t^0(a,b)+\delta_1 t(t\frac{d}{dt}+A_0-\tilde{C})]v_1(t)=0. \tag{3.5}$$

Remark 3.2. Equation (3.5) is obtained from Equation (3.1) by replacing $\tilde{C}$ by $A_0-\tilde{C}$, $\delta_1 t$ by $-\delta_1 t$, and u by v_1, and hence we obtain the following solutions of Equation (3.5) from those of Equation (3.1) which are given in Lemma 3.1:

$$\psi_{-\alpha}(\delta_1 t):=(\delta_1 t)^{-\alpha}\cdot{}_1F_1(A_0-\tilde{C}-\alpha;A_0-2\alpha;-\delta_1 t),\quad \alpha=a,\ b, \tag{3.6}$$

$$\tilde{\psi}_{-a}(\delta_1 t)=\tilde{\psi}_{-b}(\delta_1 t):=\tilde{\psi}_{2,A_0-\tilde{C}}(\frac{x}{\delta_1}). \tag{3.7}$$

Here $\tilde{\psi}_{-\alpha}(\delta_1 t)$ is given by the equation which is obtained from Equation (3.6), by replacing ψ by $\tilde{\psi}$, and ${}_1F_1$ by U, as in Notation 2.1.

3.1.2 Solutions of Equations (3.1), (3.2) and (3.4)

We now present a lemma which is obtained by replacing the roles of Equations (3.1) and (3.5) in Lemma 3.3, and hence replacing δ_1 by $-\delta_1$, $\tilde{C}$ by $A_0-\tilde{C}$, and u by v_1.

Lemma 3.4. *When* $v_1(t)$ *is a solution of Equation* (3.5), $u(t)=e^{\delta_1 t}v_1(t)$ *is that of Equation* (3.1).

Lemma 3.5. *Lemma 3.1 and Lemma 3.4 with Remark 3.2 show that we have the following eight solutions of Equation* (3.1) :

$$\phi_{-\alpha}(\delta_1 t)=e^{\delta_1 t}\psi_{-\alpha}(\delta_1 t),\quad \alpha=a,\ b;\quad \tilde{\phi}_{-a}(\delta_1 t)=\tilde{\phi}_{-b}(\delta_1 t):=\tilde{\phi}_{2,\tilde{C}}(\frac{x}{\delta_1}),$$
$$e^{\delta_1 t}\tilde{\psi}_{-a}(\delta_1 t)=e^{\delta_1 t}\tilde{\psi}_{-b}(\delta_1 t):=e^{\delta_1/x}\tilde{\psi}_{2,A_0-\tilde{C}}(\frac{x}{\delta_1}). \tag{3.8}$$

Remark 3.3. The third and fourth pairs of solutions in Equation (3.8) are the asymptotic solutions of Equation (3.1), of the form of Equation (1.33), which appear when $R(\infty) = 2$, as mentioned in Remark 1.10.

Lemma 3.6. *Theorem 3.1 and Lemma 3.5 show that the functions given in Equation* (3.8), *multiplied by* $e^{\beta_1 t}$ *or* $e^{\beta_1/x}$, *are solutions of Equation* (3.4).

3.1.3 Transformation of Equation (3.1-0) via an exponential function

We put $\delta_1 = 0$ and $\delta_0 \neq 0$, we obtain the following theorem from Theorem 3.1.

Theorem 3.2. *Let* $u(t)$ *be a solution of Equation* (3.1-0), $\beta_1 \in \mathbb{C}\backslash\{0\}$, *and Equation* (3.4-0) *denote* (3.4) *in which* A_1, B_1 *and* B_2 *are given by* $A_1 = 2\beta_1$, $B_1 = A_0\beta_1 + \delta_0$ *and* $B_2 = \beta_1^2$, *which satisfy* $A_1^2 = 4B_2$. *Then* $v(t) = e^{\beta_1 t}u(t)$ *is a solution of Equation* (3.4-0).

Lemma 3.7. *Theorem 3.2 and Lemma 3.2 show that the functions given in Equation* (2.5), *multiplied by* $e^{\beta_1 t}$, *are solutions of Equation* (3.4-0).

3.2 Eight solutions of Kummer's differential equation (1.19)

When $b = 0$, $\delta_1 = 1$, $\delta_0 = 0$, $\tilde{c} = \tilde{a}$ and $a = c - 1$, $A_0 = c$, $\tilde{C} = \tilde{a}$, and Equation (3.1) becomes Kummer's differential equation (1.19), and hence Lemma 3.5 shows that we have the following solutions of Equation (1.19):

$$\phi_0(t) = e^t\psi_0(t), \quad \phi_{1-c}(t) = e^t\psi_{1-c}(t), \quad \tilde{\phi}_0(t) = \tilde{\phi}_{1-c}(t), \quad e^t\tilde{\psi}_0(t) = e^t\tilde{\psi}_{1-c}(t), \tag{3.9}$$

where

$$\phi_0(t) := {}_1F_1(\tilde{a}; c; t), \quad \phi_{1-c}(t) := t^{1-c} \cdot {}_1F_1(\tilde{a} + 1 - c; 2 - c; t), \tag{3.10}$$
$$\psi_0(t) := {}_1F_1(c - \tilde{a}; c; t), \quad \psi_{1-c}(t) := t^{1-c} \cdot {}_1F_1(1 - \tilde{a}; 2 - c; t), \tag{3.11}$$

and $\tilde{\phi}_0(t)$, $\tilde{\phi}_{1-c}(t)$, $\tilde{\psi}_0(t)$ and $\tilde{\psi}_{1-c}(t)$ are given with the aid of Equations (3.10) and (3.11) as in Notation 2.1 and Remark 3.2.

These solutions of Equation (1.19) are called Kummer's eight solutions; see Equations 13.1.12∼14 in [4].

Remark 3.4. The last pair of solutions in Equation (3.9) are the asymptotic solutions of Equation (1.19), which are solutions of the form of Equation (1.33), mentioned in Remark 1.10.

3.3 Whittaker's differential equation

We write Whittaker's differential equation as

$$t^2 \cdot \frac{d^2}{dt^2}W(t) + (\frac{1}{4} - \mu^2 + \kappa t - \frac{1}{4}t^2)W(t) = 0, \tag{3.12}$$

see Section 16.1 of [13], Section 7.1 of [3], and Equation 13.1.31 of [4]. We see that this equation is Equation (3.4) in which $v = W$, and

$$1 + a + b = 0, \quad ab = \frac{1}{4} - \mu^2, \quad A_1 = 0, \quad B_1 = -\kappa, \quad B_2 = -\frac{1}{4}. \tag{3.13}$$

We now consider a, b, A_0, δ_1, δ_0, β_1, $\tilde{C}$ and $\tilde{c}$, which are given by $\delta_0 = 0$ and

$$a = -\mu - \frac{1}{2}, \quad b = \mu - \frac{1}{2}, \quad A_0 = 0, \quad \delta_1 = 1, \quad \beta_1 = -\frac{1}{2}, \quad \tilde{C} = \tilde{c} = -\kappa, \tag{3.14}$$

so that Equations (3.3) and (3.13) are satisfied. In using the solutions of Equation (3.1) given by Equation (3.8), we use $\phi_{\kappa,-\alpha}$ in place of $\phi_{-\alpha}$ given in Equation (2.3) and $\phi_{-\kappa,-\alpha}$ in place of $\psi_{-\alpha}$ given in Equation (3.6).

Lemma 3.6 shows that the equations in Equation (3.8), multiplied by $e^{\beta_1 t}$, give solutions of Equation (3.12). In fact, if $2\mu \notin \mathbb{Z}$, the first two pairs of solutions in Equation (3.8) then give Whittaker's functions:

$$M_{\kappa,\mu}(t) := e^{-t/2}\phi_{\kappa,\mu+\frac{1}{2}}(t) = M_{-\kappa,\mu}(t) := e^{t/2}\phi_{-\kappa,\mu+\frac{1}{2}}(t), \tag{3.15}$$

where

$$\phi_{\pm\kappa,\mu+\frac{1}{2}}(t) := t^{\mu+\frac{1}{2}} \cdot {}_1F_1(\mp\kappa + \mu + \frac{1}{2}; 2\mu + 1; \pm t), \tag{3.16}$$

see Section 16.10 of [13]; and the last two pairs of solutions there give

$$W_{\pm\kappa,\mu}(t) := e^{-t/2}\tilde{\phi}_{\pm\kappa,\mu+\frac{1}{2}}(t) = W_{\pm\kappa,-\mu}(t) := e^{-t/2}\tilde{\phi}_{\pm\kappa,-\mu+\frac{1}{2}}(t), \tag{3.17}$$

where $\tilde{\phi}_{\pm\kappa,\mu+\frac{1}{2}}(t)$ are given with the aid of Equation (3.16) as in Notation 2.1 and Remark 3.2, see Section 16.3 of [13], and Equations 13.1.32 and 13.1.33 in [4].

3.4 Parabolic cylinder function

The differential equation satisfied by the parabolic cylinder functions is

$$(\frac{d^2}{dx^2} \mp \frac{1}{4}x^2 - \tilde{a})w(x) = 0, \tag{3.18}$$

see Chapter VI, Section 4 in [3], Chapter 19 in [4].

We put $t = x^2$ and $v(t) = w(x)$. Then we have the following equation for $v(t)$:

$$(4t \cdot \frac{d^2}{dt^2} + 2\frac{d}{dt} \mp \frac{1}{4}t - \tilde{a})v(t) = 0. \tag{3.19}$$

This equation can be regarded as Equation (3.4), in which $a = 0$, $b = -\frac{1}{2}$, $\delta_0 = 0$, and

$$A_1 = 2\beta_1 + \delta_1 = 0, \quad B_1 = (1 + b)\beta_1 + \tilde{c}\delta_1 = \frac{1}{4}\tilde{a}, \quad B_2 = \beta_1\delta_1 + \beta_1^2 = \mp\frac{1}{16}. \tag{3.20}$$

The parameters a, b, A_0, δ_0, δ_1, β_1. $\tilde{C}$ and $\tilde{c}$ which satisfy these, are given by $\delta_0 = 0$ and

$$a = 0, \quad b = -\frac{1}{2}, \quad A_0 = \frac{1}{2}, \quad \delta_1 = \frac{1}{2}q_\pm, \quad \beta_1 = -\frac{1}{2}\delta_1 = -\frac{1}{4}q_\pm, \quad \tilde{C} = \tilde{c} = \frac{1}{4} + \frac{\tilde{a}}{2q_\pm}, \tag{3.21}$$

where q_+ represents 1 or -1, and q_- represents i or $-i$. The solutions of Equation (3.19), given in Lemma 3.6, are

$$\begin{aligned} &e^{-\delta_1 t/2}\phi_0(\delta_1 t) = e^{\delta_1 t/2}\psi_0(\delta_1 t), \quad e^{-\delta_1 t/2}\phi_{1/2}(\delta_1 t) = e^{\delta_1 t/2}\psi_{1/2}(\delta_1 t), \\ &e^{\delta_1 t/2}\tilde{\psi}_0(\delta_1 t) = e^{\delta_1 t/2}\tilde{\psi}_{1/2}(\delta_1 t), \quad e^{-\delta_1 t/2}\tilde{\phi}_0(\delta_1 t) = e^{-\delta_1 t/2}\tilde{\phi}_{1/2}(\delta_1 t), \end{aligned} \tag{3.22}$$

where $\delta_1 = \frac{1}{2}q_\pm$,

$$\phi_0(\delta_1 t) := {}_1F_1(\frac{1}{4} + \frac{\tilde{a}}{2q_\pm}; \frac{1}{2}; \delta_1 t), \quad \phi_{1/2}(\delta_1 t) := (\delta_1 t)^{1/2} \cdot {}_1F_1(\frac{3}{4} + \frac{\tilde{a}}{2q_\pm}; \frac{3}{2}; \delta_1 t), \tag{3.23}$$

$$\psi_0(\delta_1 t) := {}_1F_1(\frac{1}{4} - \frac{\tilde{a}}{2q_\pm}; \frac{1}{2}; \delta_1 t), \quad \psi_{1/2}(\delta_1 t) := (\delta_1 t)^{1/2} \cdot {}_1F_1(\frac{3}{4} - \frac{\tilde{a}}{2q_\pm}; \frac{3}{2}; \delta_1 t), \tag{3.24}$$

and $\tilde{\phi}_0(\delta_1 t)$, $\tilde{\phi}_{1/2}(\delta_1 t)$, $\tilde{\psi}_0(\delta_1 t)$ and $\tilde{\psi}_{1/2}(\delta_1 t)$ are given with the aid of Equation (3.23) as in Notation 2.1 and Remark 3.2.

The corresponding solutions of Equation (3.18) are obtained from Equation (3.22), by replacing t by x^2. The first three solutions thus obtained, for $q_+ = 1$, are given in Equations 19.2.1∼19.2.4, 19.8.1 and 19.8.2 in [4] for $\tilde{a} = a$.

3.5 Transformation of Equation (3.1) via a product of two exponential functions

Theorem 3.3. *Let $u(t)$ be a solution of Equation* (3.1), $\beta_1 \in \mathbb{C}$, $\beta_2 \in \mathbb{C}\backslash\{0\}$, *and*

$$\begin{aligned} A_1 &= 2\beta_1 + \delta_1, \quad A_2 = -4\beta_2, \quad B_1 = A_0\beta_1 + \tilde{C}\delta_1, \\ B_2 &= \beta_1^2 + \beta_1\delta_1 - 2(1 + A_0)\beta_2, \quad B_3 = 2(2\beta_1 + \delta_1)\beta_2, \quad B_4 = 4\beta_2^2, \end{aligned} \tag{3.25}$$

so that

$$\frac{B_3}{2A_1} = \frac{B_4}{A_2} = -\beta_2, \quad A_1^2 - 2[2B_2 - (1 + A_0)A_2] = \delta_1^2 \neq 0, \tag{3.26}$$

are satisfied. Then $w(t) = e^{\beta_2 t^2} e^{\beta_1 t} u(t)$ is a solution of

$$[{}_2D_t^0(a, b) - t(A_1 t\frac{d}{dt} + B_1) + t^2(A_2 t\frac{d}{dt} + B_2) + B_3 t^3 + B_4 t^4]w(t) = 0. \tag{3.27}$$

This fact and Lemma 3.4 show that the functions given in Equation (3.8), *multiplied by $e^{\beta_2 t^2} e^{\beta_1 t}$, are solutions of Equation* (3.27).

A proof of this theorem is given in Section 3.7.

Remark 3.5. By using Lemmas 1.2 and 1.3 for Equation (3.27), we obtain $R(0) = 1$, and $\tilde{K}_1(0) = 2$, $\tilde{K}_2(0) = 4$, and hence $R(\infty) = \tilde{R}(0) = 2$. This shows that we have the same singularities for this equation as for Equations (3.1) and (3.4).

In the following, we use notations (3.27-1) and (3.27-0), as in Notation 3.1.

3.5.1 Transformation of Equation (3.1-0) via a product of two exponential functions

Theorem 3.4. *Let $u(t)$ be a solution of Equation* (3.1-0), $\beta_1 \in \mathbb{C}$ *and* $\beta_2 \in \mathbb{C}\backslash\{0\}$, *so that*

$$\frac{B_3}{2A_1} = \frac{B_4}{A_2} = -\beta_2, \quad A_1^2 - 2[2B_2 - (1 + A_0)A_2] = 0, \tag{3.28}$$

are satisfied. Then $w(t) = e^{\beta_2 t^2} e^{\beta_1 t} u(t)$ is a solution of Equation (3.27-0), *and hence the functions given in Equation* (2.5), *multiplied by $e^{\beta_2 t^2} e^{\beta_1 t}$, are solutions of Equation* (3.27-0).

See Theorem 3.3 for the case in which the second equation in Equation (3.28) is not satisfied.

3.6 Transformation of Equations (3.1) and (3.1-0) via a function

Theorem 3.5. *Let $u(t)$ be a solution of Equation* (3.1), *and $\beta \in \mathbb{C}\backslash\{0\}$. Then $v(t) = (1 - \rho t)^\beta u(t)$ is a solution of*

$$\begin{aligned} {}_2D_t^0(a, b)v(t) + [2\rho\beta(1 - \rho t)^{-1} - \delta_1]t \cdot t\frac{d}{dt}v(t) + [\rho^2\beta(1 + \beta)t^2(1 - \rho t)^{-2} \\ + \rho\beta(A_0 - \delta_1 t)t(1 - \rho t)^{-1} - \delta_1\tilde{C}t]v(t) = 0. \end{aligned} \tag{3.29}$$

This fact and Lemma 3.5 show that solutions of Equation (3.29) *are obtained from the first two pairs of functions given in Equation* (3.8), *by multiplying $(1 - \rho t)^\beta$, and from the last two pairs of functions, by multiplying $(\rho t)^\beta(1 - \frac{1}{\rho t})^\beta$.*

In the following, we use notations (3.29-1) *and* (3.29-0), *as in Notation 3.1.*

A proof of this theorem is given in Section 3.8.

Lemma 3.8. *Theorem 3.5 and Lemma 3.2 show that the functions given in Equation* (2.5)*, multiplied by* $(1-\rho t)^{\beta}$*, are solutions of Equation* (3.29-0)*.*

By putting $\beta = -1$ in (3.29) and then multiplying it by $(1-\rho t)$, we obtain

Lemma 3.9. *Let $u(t)$ be a solution of Equation* (3.1)*, and*

$$A_1 = 2\rho + \delta_1, \quad B_1 = \rho A_0 + \delta_1 \tilde{C}, \quad A_2 = \delta_1 \rho, \quad B_2 = \rho\delta_1(1+\tilde{C}). \tag{3.30}$$

Then $v(t) = (1-\rho t)^{-1} u(t)$ is a solution of

$$[(1-\rho t)\cdot {}_2D_t^0(a,b) - t(A_1 \cdot t\frac{d}{dt} + B_1) + t^2(A_2 \cdot t\frac{d}{dt} + B_2)]v(t) = 0. \tag{3.31}$$

The solution of Equation (3.31) *are given by those of Equation* (3.29) *in Theorem 3.5 for $\beta = -1$.*

Lemma 3.10. *Let $u(t)$ be a solution of Equation* (3.1-0)*, and Equation* (3.31-0) *be Equation* (3.31)*, in which $A_1 = 2\rho$, $B_1 = \rho A_0 + \delta_0$, $A_2 = 0$ and $B_2 = \rho\delta_0$. Then $v(t) = (1-\rho t)^{-1}u(t)$ is a solution of Equation* (3.31-0)*, and hence the functions given in Equation* (2.5)*, multiplied by $(1-\rho t)^{-1}$, are solutions of Equation* (3.31-0)*.*

Remark 3.6. In this case, the original Equation (3.1) has one regular singularity at 0 and one irregular one at ∞, but the transformed Equations (3.29) and (3.31) have two regular singularities at 0 and 1, as shown by Remark 1.5, and one irregular one at ∞, with $R(\infty) = \tilde{R}(0) = 2$, since Lemma 1.3 shows that $\tilde{K}_1(0) = 2$ and $\tilde{K}_2(0) = 3$. We note that the transformed Equation (3.31) is an example of the differential equation, of which Stewart [14] discussed the asymptotic form of the solution around ∞. In the present example, we have explicit expressions of the solutions given in Lemma 3.9 for $\beta = -1$.

3.7 Proofs of Theorems 3.1 and 3.3

Theorems 3.1 and 3.3 are proved with the aid of the following lemma.

Lemma 3.11. *Let $v(t) = e^{\beta_n t^n} u(t)$. Then*

$$t\frac{d}{dt}u(t) = t\frac{d}{dt}[e^{-\beta_n t^n} v(t)] = te^{-\beta_n t^n}[\frac{d}{dt}v(t) - n\beta_n t^{n-1} v(t)], \tag{3.32}$$

$$t^2\frac{d^2}{dt^2}u(t) = t^2\frac{d^2}{dt^2}[e^{-\beta_n t^n} v(t)] = t^2 e^{-\beta_n t^n}[\frac{d^2}{dt^2}v(t) - 2n\beta_n t^{n-1}\frac{d}{dt}v(t) - n(n-1)\beta_n t^{n-2} v(t) + n^2\beta_n^2 t^{2n-2} v(t)]. \tag{3.33}$$

By using these, with Equation (1.11)*, we obtain*

$$\begin{aligned} e^{\beta_n t^n}\,{}_2\mathcal{D}_t^0(a,b)u(t) &= e^{\beta_n t^n}[t^2\frac{d^2}{dt^2} + A_0 \cdot t\frac{d}{dt} + ab]u(t) \\ &= [{}_2\mathcal{D}_t^0(a,b) - 2n\beta_n t^n \cdot t\frac{d}{dt} - n(n-1+A_0)\beta_n t^n + n^2\beta_n^2 t^{2n}]v(t), \end{aligned} \tag{3.34}$$

$$-\delta_1 t e^{\beta_n t^n}\,{}_1\mathcal{D}_t^0(\tilde{C})u(t) = -\delta_1 t(t\frac{d}{dt} - n\beta_n t^n + \tilde{C})v(t). \tag{3.35}$$

Lemma 3.12. *When $n = 1$, by using Equations* (3.34) *and* (3.35)*, we have*

$$\begin{aligned} &e^{\beta_1 t}[{}_2\mathcal{D}_t^0(a,b) - \delta_1 t \cdot {}_1\mathcal{D}_t^0(\tilde{C})]u(t) \\ &= [{}_2D_t^0(a,b) - (2\beta_1 + \delta_1)t \cdot t\frac{d}{dt} - (\beta_1 A_0 + \delta_1\tilde{C})t + \beta_1(\beta_1+\delta_1)t^2]v(t) = 0. \end{aligned} \tag{3.36}$$

Proof of Theorem 3.1. When $u(t)$ satisfies (3.1), Equation (3.36) is satisfied, and the equation satisfied by $v(t) = e^{\beta_1 t}u(t)$ is given by Equation (3.36). □

Lemma 3.13. *By using Equation* (3.36)*, with Equations* (3.34) *and* (3.32) *for* $n = 2$*, we have*

$$e^{\beta_2 t^2} e^{\beta_1 t}[{}_2\mathcal{D}_t^0(a,b) - \delta_1 t \cdot {}_1\mathcal{D}_t^0(\tilde{C})]u(t)$$
$$= [{}_2D_t^0(a,b) - ((2\beta_1 + \delta_1)t + 4\beta_2 t^2)t\frac{d}{dt} - (\beta_1 A_0 + \delta_1 \tilde{C})t$$
$$+(\beta_1(\beta_1 + \delta_1) - 2(1 + A_0)\beta_2)t^2 + 2\beta_2(2\beta_1 + \delta_1)t^3 + 4\beta_2^2 t^4]w(t) = 0, \tag{3.37}$$

where the terms involving β_2 *in Equation* (3.37) *are added to the terms in Equation* (3.36)*.*

Proof of Theorem 3.3. When $u(t)$ satisfies (3.1), Equation (3.37) is satisfied, and the equation satisfied by $w(t) = e^{\beta_2 t^2} e^{\beta_1 t} u(t)$ is given by Equation (3.37). □

3.8 Proof of Theorem 3.5

Theorem 3.5 is proved with the aid of the following lemma.

Lemma 3.14. *Let* $\beta \in \mathbb{C}\backslash\{0\}$ *and* $v(t) = (1 - \rho t)^{\beta} u(t)$*. Then*

$$t\frac{d}{dt}u(t) = t\frac{d}{dt}[(1 - \rho t)^{-\beta} v(t)] = t(1 - \rho t)^{-\beta}[\frac{d}{dt}v(t) + \rho\beta(1 - \rho t)^{-1} v(t)], \tag{3.38}$$

$$t^2\frac{d^2}{dt^2}u(t) = t^2\frac{d^2}{dt^2}[(1 - \rho t)^{-\beta} v(t)] = t^2(1 - \rho t)^{-\beta}[\frac{d^2}{dt^2}v(t)$$
$$+2\rho\beta(1 - \rho t)^{-1}\frac{d}{dt}v(t) + \rho^2\beta(\beta + 1)(1 - \rho t)^{-2} v(t)]. \tag{3.39}$$

By using these, with Equation (1.11)*, we have*

$$(1 - \rho t)^{\beta}{}_2\mathcal{D}_t^0(a,b)u(t) = (1 - \rho t)^{\beta}[t^2 \cdot \frac{d^2}{dt^2} + A_0 \cdot t \cdot \frac{d}{dt} + ab]u(t)$$
$$= {}_2\mathcal{D}_t^0(a,b)v(t) + 2\rho\beta(1 - \rho t)^{-1} t \cdot t\frac{d}{dt}v(t)$$
$$+[\rho^2\beta(\beta + 1)(1 - \rho t)^{-2} t^2 + A_0\rho\beta(1 - \rho t)^{-1} t]v(t), \tag{3.40}$$

$$(1 - \rho t)^{\beta}{}_1\mathcal{D}_t^0(c)u(t) = (1 - \rho t)^{\beta}[t\frac{d}{dt} + c]u(t) = t\frac{d}{dt}v(t) + [\rho\beta(1 - \delta t)^{-1} t + c]v(t). \tag{3.41}$$

Proof of Theorem 3.5. Let $v(t) = (1 - \rho t)^{\beta} u(t)$. When $u(t)$ satisfies (3.1),

$$(1 - \rho t)^{\beta}[{}_2\mathcal{D}_t^0(a,b) - \delta_1 t \cdot {}_1\mathcal{D}_t^0(\tilde{c}) - \delta_0 t \cdot {}_0\mathcal{D}_t^0]u(t) = 0, \tag{3.42}$$

is satisfied. Then by using Equations (3.40) and (3.41) in Equation (3.42), we obtain an equation which is satisfied by $v(t)$, that is Equation (3.29). □

4 Transformations of Equations (1.10) and (1.16)

Equations (1.10) and (1.16) are expressed as follows:

$$[\delta_1 \cdot {}_1\mathcal{D}_t^0(C) + t \cdot {}_2\mathcal{D}_t^0(\tilde{a},\tilde{b})]u(t) = \delta_1(t\frac{d}{dt} + C)u(t) + t[t^2\frac{d^2}{dt^2} + (2 - \tilde{A}_0)t\frac{d}{dt} + \tilde{a}\tilde{b}]u(t) = 0, \tag{4.1}$$

$$[{}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{b}) - \delta_1 x \cdot {}_1\mathcal{D}_x^0(-C)]\tilde{u}(x) = [x^2\frac{d^2}{dx^2} + \tilde{A}_0 \cdot x\frac{d}{dx} + \tilde{a}\tilde{b} - \delta_1 x(x\frac{d}{dx} - C)]\tilde{u}(x) = 0, \tag{4.2}$$

where $\tilde{A}_0 = 1 - \tilde{a} - \tilde{b}$ and $C = c - \frac{\delta_0}{\delta_1}$. This equation has an irregular and a regular singular point at $t = 0$ and $t = \infty$, respectively, as mentioned in Remarks 1.6 and 1.9.

In the following, we use notations (4.2-1) and (4.2-0), as in Notation 3.1.

As stated in Remark 1.2, Equation (1.16) is obtained from Equation (1.9), by replacing variable t by x and parameters. In particular,

Remark 4.1. Equation (4.2) is obtaind from Equation (3.1), by replacing t by x, $u(t)$ by $\tilde{u}(x)$, a by $-\tilde{a}$, b by $-\tilde{b}$, $\tilde{c}$ by $-c$, A_0 by $\tilde{A}_0$, and $\tilde{C}$ by $-C$. As a consequence, the results for the former equation, are obtained from the corresponding results for the latter, which are given in Section 3. We note here that when a replacement of t by x occurs, replacements of ϕ by $\tilde{\phi}$, and of ψ by $\tilde{\psi}$ occur.

Lemma 4.1. *By Theorem 2.1(iv) and Lemma 2.4, if $a - b \notin \mathbb{Z}$ and $\delta_1 \neq 0$, Equations (4.2) and (4.1) have the solutions given by Equations (2.17) and (2.7), respectively.*

Lemma 4.2. *By Theorem 2.1(v), if $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, $\delta_1 = 0$ and $\delta_1 \neq 0$, Equation (4.2-0) has the solution given by Equation (2.8).*

4.1 Transformations of Equations (4.2) and (4.1) via an exponential function

By Remark 4.1, we have the following theorem from Theorem 3.1.

Theorem 4.1. *Let $\tilde{u}(x)$ and $u(t)$ be solutions of Equations (4.2) and (4.1), respectively, $\beta_1 \in \mathbb{C}\backslash\{0\}$, and A_1, B_1 and B_2 be obtained from Equation (3.3), by replacing A_0 by $\tilde{A}_0$, and $\tilde{C}$ by $-C$. Then $\tilde{v}(x) = e^{\beta_1 x}\tilde{u}(x)$ is a solution of*

$$[{}_2\mathcal{D}_x^0(-\tilde{a}, -\tilde{b}) - x(A_1 x\frac{d}{dx} + B_1) + B_2 x^2]\tilde{v}(x) = 0, \tag{4.3}$$

and $v(t) = e^{\beta_1/t}u(t)$ is a solution of

$$[{}_2\mathcal{D}_t^0(\tilde{a}, \tilde{b}) - \frac{1}{t}(-A_1 t\frac{d}{dt} + B_1) + B_2\frac{1}{t^2}]v(t) = 0. \tag{4.4}$$

See Theorem 4.2 in Section 4.1.3 for the case of $A_1^2 - 4B_2 = 0$.

4.1.1 Transformation of Equation (4.2) via a special exponential function

When $\delta_1 \neq 0$, we choose $\beta_1 = -\delta_1$. Then Theorem 4.1 becomes to

Lemma 4.3. *Let $\tilde{u}(x)$ be a solution of Equation (4.2). Then $\tilde{v}_1(x) = e^{-\delta_1 x}\tilde{u}(x)$ is a solution of*

$$[{}_2\mathcal{D}_x^0(-\tilde{a}, -\tilde{b}) + \delta_1 x(x\frac{d}{dx} + \tilde{A}_0 + C)]\tilde{v}_1(x) = 0. \tag{4.5}$$

Remark 4.2. Equation (4.5) is obtained from Equation (4.2) by replacing C by $-\tilde{A}_0 - C$, $\delta_1 x$ by $-\delta_1 x$, and $\tilde{u}$ by $\tilde{v}_1$, and hence we obtain the following solutions of Equation (4.5) from those of Equation (4.2) which are given by Equations (2.7) and (2.17):

$$\tilde{\psi}_{\tilde{\alpha}}(\delta_1 x) := (\delta_1 x)^{\tilde{\alpha}} \cdot {}_1F_1(\tilde{\alpha} + \tilde{A}_0 + C; \tilde{A}_0 + 2\tilde{\alpha}; -\delta_1 x), \quad \tilde{\alpha} = \tilde{a},\ \tilde{b}, \tag{4.6}$$

$$\psi_{\tilde{a}}(\delta_1 x) = \psi_{\tilde{b}}(\delta_1 x) := \psi_{2,\tilde{A}_0+C}(\frac{1}{\delta_1}t) := (\frac{1}{\delta_1}t)^{\tilde{A}_0+C} \cdot {}_2F_0(\tilde{a} + \tilde{A}_0 + C, \tilde{b} + \tilde{A}_0 + C;\, ; \frac{1}{\delta_1}t). \tag{4.7}$$

Here $\psi_{\tilde{a}}(\delta_1 t)$ and $\psi_{\tilde{b}}(\delta_1 t)$ are given by the equations which are obtained from Equation (4.6) for $\tilde{\alpha} = \tilde{a}$ and, $\tilde{\alpha} = \tilde{b}$, respectively, by replacing $\tilde{\psi}$ by ψ, and ${}_1F_1$ by U, as in Notation 2.1.

4.1.2 Solutions of Equations (4.2), (4.1), (4.3) and (4.4)

We confirm the following lemma by using Remark 4.1 and Lemma 3.1, with the aid of Theorem 2.1 (i) and (iv), Lemmas 2.3 and 2.4, and Remarks 3.2 and 4.2.

Lemma 4.4. *Lemma 4.2 shows that, if $\tilde{a}-\tilde{b} \notin \mathbb{Z}$, we have the following solutions of Equations* (4.2) *and* (4.1)*:*

$$\tilde{\phi}_{\tilde{\alpha}}(\delta_1 x) = e^{\delta_1 x}\tilde{\psi}_{\tilde{\alpha}}(\delta_1 x), \quad \tilde{\alpha} = \tilde{a},\ \tilde{b}; \tag{4.8}$$

$$\phi_{\tilde{a}}(\delta_1 x) = \phi_{\tilde{b}}(\delta_1 x) := \phi_{2,-C}(\frac{t}{\delta_1}), \quad e^{\delta_1 x}\psi_{\tilde{a}}(\delta_1 x) = e^{\delta_1 x}\psi_{\tilde{b}}(\delta_1 x) := e^{\delta_1/t}\psi_{2,\tilde{A}_0+C}(\frac{t}{\delta_1}). \tag{4.9}$$

Lemma 4.5. *Theorem 4.1 and Lemma 4.4 show that, if $\tilde{a}-\tilde{b} \notin \mathbb{Z}$, the functions given in Equations* (4.8) *and* (4.9)*, multiplied by $e^{\beta_1 x}$ or $e^{\beta_1/t}$, are solutions of Equations* (4.3) *and* (4.4)*, respectively.*

4.1.3 Transformations of Equations (4.2-0) and (4.1-0) via an exponential function

When $\delta_1 = 0$ and $\delta_0 \neq 0$, Theorem 4.1 becomes to the following theorem.

Theorem 4.2. *Let $\tilde{u}(x)$ and $u(t)$ be a solution of Equations* (4.2-0) *and* (4.1-0)*, $\beta_1 \in \mathbb{C}\backslash\{0\}$, and Equations* (4.3-0) *and* (4.4-0)*, respectively, denote* (4.3) *and* (4.4) *in which A_1, B_1 and B_2 are given by $A_1 = 2\beta_1$, $B_1 = \beta_1\tilde{A}_0+\delta_0$, and $B_2 = \beta_1^2$, which satisfy $A_1^2 = 4B_2$. Then $\tilde{v}(x) = e^{\beta_1 x}\tilde{u}(x)$ and $v(t) = e^{\beta_1/t}u(t)$ are solutions of Equations* (4.3-0) *and* (4.4-0)*, respectively.*

Lemma 4.6. *Theorem 4.2 and Lemma 4.2 show that, if $\tilde{a}-\tilde{b} \notin \mathbb{Z}$, the functions given in Equation* (2.8)*, multiplied by $e^{\beta_1 x}$, are solutions of Equation* (4.3-0)*.*

4.2 Kummer's differential equation for Equation (4.2-1)

When $\tilde{b} = 0$, $\tilde{a} = 1-\tilde{c}$ and $c = -a$, Equation (4.2-1) becomes

$$[\cdot_2\mathcal{D}_x^0(\tilde{c}-1,0) - \delta_1 x \cdot {}_1\mathcal{D}_x^0(a)]\tilde{u}(x) = 0, \tag{4.10}$$

which is obtained from Kummer's differential equation given by Equation (1.19), by replacing c by $\tilde{c}$, $\tilde{a}$ by a, t by $\delta_1 x$, and $u(t)$ by $\tilde{u}(x)$. As a consequence, we obtain Kummer's eight solutions of Equation (4.10), from Equations (3.9) and (3.10) by replacing c by $\tilde{c}$, $\tilde{a}$ by a, and t by $\delta_1 x$.

4.3 Transformations of Equations (4.2) and (4.1) via a product of two exponential functions

By Remark 4.1, we have the following theorem from Theorem 3.3.

Theorem 4.3. *Let $\tilde{u}(x)$ and $u(t)$ be solutions of Equations* (4.2) *and* (4.1)*, respectively, $\beta_1 \in \mathbb{C}\backslash\{0\}$, $\beta_2 \in \mathbb{C}\backslash\{0\}$, and A_1, A_2, and $B_1 \sim B_4$ be obtained from Equation* (3.25)*, by replacing A_0 by $\tilde{A}_0$, and $\tilde{C}$ by $-C$. Then $\tilde{w}(x) = e^{\beta_2 x^2}e^{\beta_1 x}\tilde{u}(x)$ is a solution of*

$$[{}_2\mathcal{D}_x^0(-\tilde{a},-\tilde{b}) - x(A_1 x\frac{d}{dx} + B_1) + x^2(A_2 x\frac{d}{dx} + B_2) + B_3 x^3 + B_4 x^4]\tilde{w}(x) = 0, \tag{4.11}$$

and $w(t) = e^{\beta_2/t^2}e^{\beta_1/t}u(t)$ is a solution of

$$[{}_2\mathcal{D}_t^0(\tilde{a},\tilde{b}) + \frac{1}{t}(A_1 t\frac{d}{dt} - B_1) + \frac{1}{t^2}(-A_2 t\frac{d}{dt} + B_2) + B_3\frac{1}{t^3} + B_4\frac{1}{t^4}]w(t) = 0, \tag{4.12}$$

and hence ihe functions given in Equations (4.8) *and* (4.9)*, multiplied by $e^{\beta_2 x^2}e^{\beta_1 x}$ or $e^{\beta_2/t^2}e^{\beta_1/t}$, are solutions of Equations* (4.11) *and* (4.12)*, respectively.*

In the present case, $A_1^2 - 2[2B_2 - (1+\tilde{A}_0)A_2] = \delta_1^2 \neq 0$ is satisfied. The case when this does not hold, is treated in Theorem 4.4 in Section 4.3.1.

4.3.1 Transformations of Equations (4.2-0) and (4.1-0) via a product of two exponential functions

When $\delta_1 = 0$ and $\delta_0 \neq 0$, Theorem 4.3 becomes the following theorem.

Theorem 4.4. *Let $\tilde{u}(x)$ and $u(t)$ be a solution of Equations (4.2-0) and (4.1-0), respectively, $\beta_1 \in \mathbb{C}$, $\beta_2 \in \mathbb{C}\backslash\{0\}$, and A_1, A_2, and $B_1 \sim B_4$ be obtained from Equation (3.25), by replacing A_0 by $\tilde{A}_0$. Then $\tilde{w}(x) = e^{\beta_2 x^2} e^{\beta_1 x}\tilde{u}(x)$ and $w(t) = e^{\beta_2/t^2} e^{\beta_1/t} u(t)$ are solutions of Equations (4.11-0) and (4.12-0), respectively, and hence the functions given in Equation (2.8), multiplied by $e^{\beta_2 x^2} e^{\beta_1 x}$, are solutions of Equation (4.11-0).*

5 Transformation of Equation (1.8)

We now consider Equation (1.8) for $\delta_2' = 1$, $\delta_1' = 0$, $\delta_2 = 1$, $\delta_1 = 0$, which is expressed by

$$\begin{aligned}&[{}_2D_t^0(a,b) - t \cdot {}_2D_t^0(\tilde{a},\tilde{b})]u(t)\\ &\quad = [t^2\frac{d^2}{dt^2} + A_0 \cdot t\frac{d}{dt} + ab]u(t) - t[t^2\frac{d^2}{dt^2} + (1+\tilde{a}+\tilde{b})t\frac{d}{dt} + \tilde{a}\tilde{b}]u(t) = 0,\end{aligned} \tag{5.1}$$

where $A_0 = 1 + a + b$. By putting $x = \frac{1}{t}$ and $\tilde{u}(x) = u(t)$ in this equation, by Lemma 1.1, we obtain

$$[{}_2\tilde{D}_x^0(-\tilde{a},-\tilde{b}) - x \cdot {}_2\tilde{D}_x^0(-a,-b)]\tilde{u}(x) = 0. \tag{5.2}$$

Lemma 5.1. *By Theorem 2.1(i) and its proof in Section 2.1, if $a - b \notin \mathbb{Z}$ and $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, we obtain the following solutions of Equations (5.1) and (5.2):*

$$\phi_{-\alpha}(t) := t^{-\alpha} \cdot {}_2F_1(\tilde{a}-\alpha, \tilde{b}-\alpha; A_0 - 2\alpha; t), \quad \alpha = a,\ b, \tag{5.3}$$

$$\tilde{\phi}_{\tilde{a}}(x) := x^{\tilde{a}} \cdot {}_2F_1(\tilde{a}-a, \tilde{a}-b; 1+\tilde{a}-\tilde{b}; x), \quad \tilde{\phi}_{\tilde{b}}(x) := x^{\tilde{b}} \cdot {}_2F_1(\tilde{b}-a, \tilde{b}-b; 1-\tilde{a}+\tilde{b}; x). \tag{5.4}$$

5.1 Transformation of Equation (5.1) via a function

Theorem 5.1. *Let $u(t)$ be a solution of Equation (5.1), and*

$$A = A_0 - \tilde{b}, \quad B = A_0 - \tilde{a}, \quad \beta = \tilde{a} + \tilde{b} - A_0 + C = -A - B + A_0 + C. \tag{5.5}$$

Then $v(t) = (1-t)^\beta u(t)$ is a solution of

$$\begin{aligned}&(1-t)t^2\frac{d^2}{dt^2}v(t) + [A_0 - (1+A+B-2C)t]t\frac{d}{dt}v(t)\\ &\qquad + [ab - ABt + CA_0(t + \frac{t^2}{1-t}) - C(A+B-C)\frac{t^2}{1-t}]v(t) = 0.\end{aligned} \tag{5.6}$$

Proof. Let $v(t) = (1-t)^\beta u(t)$. When $u(t)$ satisfies (5.1),

$$(1-t)^\beta[{}_2\mathcal{D}_t^0(a,b) - t \cdot {}_2\mathcal{D}_t^0(\tilde{a},\tilde{b})]u(t) = 0, \tag{5.7}$$

is satisfied. Then by using Equation (3.40) divided by $(1-t)^2$, in (5.7), we obtain

$$\begin{aligned}&(1-t)t^2\frac{d^2}{dt^2}v(t) + [A_0 - (1+\tilde{a}+\tilde{b}-2\beta)t]t\frac{d}{dt}v(t)\\ &\qquad + [ab - \tilde{a}\tilde{b}t + A_0\beta(t + \frac{t^2}{1-t}) - (\tilde{a}+\tilde{b}-\beta)\beta\frac{t^2}{1-t}]v(t) = 0,\end{aligned} \tag{5.8}$$

which is satisfied by $v(t)$. This can be rewritten as Equation (5.6). □

5.1.1 Transformation of Equation (5.1) via a special function

We now consider the case of $\beta = \tilde{a}+\tilde{b}-A_0$ in Theorem 5.1. Then we obtain the following lemma from Theorem 5.1.

Lemma 5.2. *Let $u(t)$ be a solution of Equation (5.1), and β, A and B be given by*

$$\beta = \tilde{a}+\tilde{b}-A_0, \quad A = \tilde{a}-\beta = -\tilde{b}+A_0, \quad B = \tilde{b}-\beta = -\tilde{a}+A_0. \tag{5.9}$$

Then $v_1(t) = (1-t)^{\beta}u(t)$ is a solution of

$$(1-t)t^2\frac{d^2}{dt^2}v_1(t) + [A_0-(1+A+B)t]t\frac{d}{dt}v_1(t) + (ab-ABt)v_1(t) = 0. \tag{5.10}$$

Remark 5.1. Equation (5.10) is obtained from Equation (5.1), by replacing $\tilde{a}$ by A, $\tilde{b}$ by B, and u by v_1. By using Lemma 5.1, we have solutions of Equation (5.10), given by

$$\psi_{-\alpha}(t) := t^{-\alpha}\cdot {}_2F_1(-\tilde{a}+A_0-\alpha, -\tilde{b}+A_0-\alpha; A_0-2\alpha; t) = (1-t)^{\beta}\phi_{-\alpha}(t), \quad \alpha = a,\ b, \tag{5.11}$$

$$\tilde{\psi}_{-\tilde{a}+A_0}(x) := x^{-\tilde{a}+A_0}\cdot {}_2F_1(-\tilde{a}+A_0-a, -\tilde{a}+A_0-b; 1-\tilde{a}+\tilde{b}; x) = x^{-\beta}(1-x)^{\beta}\tilde{\phi}_{\tilde{b}}(x), \tag{5.12}$$

and the equation which is obtained from Equation (5.12) by exchanging $\tilde{a}$ and $\tilde{b}$, where $\beta = \tilde{a}+\tilde{b}-A_0$.

5.1.2 Solutions of Equation (5.1)

We now present a lemma which is obtained by replacing the roles of Equations (5.1) and (5.10), and hence replacing $\tilde{a}$ by A, $\tilde{b}$ by B, u by v_1, and β by $-\beta$.

Lemma 5.3. *Let $v_1(t)$ be a solution of Equation (5.10), and β, $\tilde{a}$ and $\tilde{b}$ be given by Equation (5.9), so that*

$$-\beta = A+B-A_0, \quad \tilde{a} = A+\beta, \quad \tilde{b} = B+\beta. \tag{5.13}$$

Then $u(t) = (1-t)^{-\beta}v_1(t)$ is a solution of Equation (5.1).

Lemma 5.4. *Let $\beta = \tilde{a}+\tilde{b}-A_0$, and $\phi_{-\alpha}(t)$, $\tilde{\phi}_{-\tilde{a}}(x)$ and $\tilde{\phi}_{-\tilde{b}}(x)$ be given in Lemma 5.1, and $\psi_{-\alpha}(t)$, $\tilde{\psi}_{-\tilde{a}+A_0}(x)$ and $\tilde{\psi}_{-\tilde{b}+A_0}(x)$ be given in Remark 5.1. Then Lemma 5.3 shows that we have the following solutions of Equation (5.1):*

$$\phi_{-\alpha}(t) = (1-t)^{-\beta}\psi_{-\alpha}(t), \quad \alpha = a,\ b, \tag{5.14}$$

$$\tilde{\phi}_{\tilde{a}}(x) = x^{\beta}(1-x)^{-\beta}\tilde{\psi}_{-\tilde{b}+A_0}(x), \quad \tilde{\phi}_{\tilde{b}}(x) = x^{\beta}(1-x)^{-\beta}\tilde{\psi}_{-\tilde{a}+A_0}(x). \tag{5.15}$$

5.2 24 solutions of the hypergeometric differential equation

When $b = 0$ and $c = 1+a$, $a = c-1$, $A_0 = c$, and Equation (5.1) becomes the hypergeometric differential equation:

$$\begin{aligned}[{}_2D_t^0(c-1,0) - t\cdot {}_2D_t^0(\tilde{a},\tilde{b})]u(t) &= (t^2\frac{d^2}{dt^2}+ct\frac{d}{dt})u(t) - t[t^2\frac{d^2}{dt^2}+(1+\tilde{a}+\tilde{b})t\frac{d}{dt}+\tilde{a}\tilde{b}]u(t)\\ &= 0.\end{aligned} \tag{5.16}$$

By putting $x = \frac{1}{t}$ and $\tilde{u}(x) = u(t)$ in Equation (5.16), by Lemma 1.1, we obtain

$$\begin{aligned}&[{}_2\tilde{D}_x^0(-\tilde{a},-\tilde{b}) - x\cdot {}_2\tilde{D}_x^0(1-c,0)]\tilde{u}(x)\\ &= [x^2\frac{d^2}{dx^2}+\tilde{A}_0x\frac{d}{dx}+\tilde{a}\tilde{b}]\tilde{u}(x) - x[x^2\frac{d^2}{dx^2}+(2-c)x\frac{d}{dx}]\tilde{u}(x) = 0.\end{aligned} \tag{5.17}$$

With the aid of Lemma 5.4, we obtain the following solutions of Equations (5.16) and (5.17):

$$\phi_0(t) := {}_2F_1(\tilde{a}, \tilde{b}; c; t) = (1-t)^{c-\tilde{a}-\tilde{b}} \cdot {}_2F_1(c-\tilde{a}, c-\tilde{b}; c; t), \tag{5.18}$$

$$\begin{aligned}\phi_{1-c}(t) &:= t^{1-c} \cdot {}_2F_1(1+\tilde{a}-c, 1+\tilde{b}-c; 2-c; t) \\ &= (1-t)^{c-\tilde{a}-\tilde{b}} t^{1-c} \cdot {}_2F_1(1-\tilde{a}, 1-\tilde{b}; 2-c; t),\end{aligned} \tag{5.19}$$

$$\begin{aligned}\tilde{\phi}_{\tilde{a}}(x) &:= x^{\tilde{a}} \cdot {}_2F_1(\tilde{a}+1-c, \tilde{a}; 1+\tilde{a}-\tilde{b}; x) \\ &= (1-x)^{c-\tilde{a}-\tilde{b}} x^{\tilde{a}} \cdot {}_2F_1(-\tilde{b}+1, -\tilde{b}+c; , 1-\tilde{a}+\tilde{b}; x),\end{aligned} \tag{5.20}$$

and the equation which is obtained from Equation (5.20) by exchanging $\tilde{a}$ and $\tilde{b}$, These are eight of Kummer's 24 solutions of Equation (5.16); see Equations 15.5.3-14 in [4], and [15].

We put $t = 1-\eta$ and $u(t) = y(\eta)$ in Equation (5.16). Then we obtain

$$\begin{aligned}&[{}_2\tilde{D}^0_\eta(\tilde{a}+\tilde{b}-c, 0) - \eta \cdot {}_2\tilde{D}^0_\eta(\tilde{a}, \tilde{b})]y(\eta) \\ &= [\eta^2 \frac{d^2}{d\eta^2} + (1+\tilde{a}+\tilde{b}-c)\eta\frac{d}{d\eta}]y(\eta) - \eta[\eta^2\frac{d^2}{d\eta^2} + (1+\tilde{a}+\tilde{b})\eta\frac{d}{d\eta} + \tilde{a}\tilde{b}]y(\eta) = 0.\end{aligned} \tag{5.21}$$

This equation is obtained from (5.16), by replacing c by $1+\tilde{a}+\tilde{b}-c$, t by η, and u by y, and hence with the aid of Equations (5.18)∼(5.20), we obtain the following solutions of Equation (5.21):

$${}_2F_1(\tilde{a}, \tilde{b}; 1+\tilde{a}+\tilde{b}-c; \eta) = (1-\eta)^{1-c} \cdot {}_2F_1(1+\tilde{b}-c, 1+\tilde{a}-c; 1+\tilde{a}+\tilde{b}-c; \eta), \tag{5.22}$$

$$\eta^{c-\tilde{a}-\tilde{b}} \cdot {}_2F_1(c-\tilde{b}, c-\tilde{a}; \tilde{A}_0+c; \eta) = (1-\eta)^{1-c}\eta^{c-\tilde{a}-\tilde{b}} \cdot {}_2F_1(1-\tilde{a}, 1-\tilde{b}; \tilde{A}_0+c; \eta), \tag{5.23}$$

$$\eta^{-\tilde{a}} \cdot {}_2F_1(c-\tilde{b}, \tilde{a}; 1+\tilde{a}-\tilde{b}; \eta^{-1}) = (1-\eta^{-1})^{1-c}\eta^{-\tilde{a}} \cdot {}_2F_1(1-\tilde{b}, 1-c+\tilde{a}; 1+\tilde{a}-\tilde{b}; \eta^{-1}), \tag{5.24}$$

and the equation which is obtained from Equation (5.24) by exchanging $\tilde{a}$ and $\tilde{b}$, By putting $\eta = 1-t$ in these equations, we obtain eight of Kummer's 24 solutions of Equation (5.16).

When we put $\tilde{u}(x) = x^{\tilde{a}}\tilde{w}(x)$, Lemma 2.2 shows that $\tilde{w}(x)$ satisfies

$$\begin{aligned}[{}_2\tilde{D}^0_x(0, \tilde{a}-\tilde{b}) - x \cdot {}_2\tilde{D}^0_x(\tilde{a}+1-c, \tilde{a})]\tilde{w}(x) = [x^2\frac{d^2}{dx^2} + (1+\tilde{a}-\tilde{b})x\frac{d}{dx}]\tilde{w}(x) \\ -x[x^2\frac{d^2}{dx^2} + (2-c+2\tilde{a})x\frac{d}{dx} + \tilde{a}(1-c+\tilde{a})]\tilde{w}(x) = 0.\end{aligned} \tag{5.25}$$

Remark 5.2. We confirm that Equation (5.25) is obtained from Equation (5.16), by replacing t by x, u by $\tilde{w}$, c by $1+\tilde{a}-\tilde{b}$, and $\tilde{b}$ by $1-c+\tilde{a}$.

The solutions of Equation (5.25), which correspond to Equations (5.22)∼(5.24), are obtained from these by the replacements stated in Remark 5.2, and the replacement of η by ζ, where $\zeta = 1-x = \frac{t-1}{t}$. They are

$${}_2F_1(\tilde{a}, 1-c+\tilde{a}; 1-c+\tilde{a}+\tilde{b}; \zeta) = (1-\zeta)^{-\tilde{a}+\tilde{b}} \cdot {}_2F_1(1-c+\tilde{b}, \tilde{b}; 1-c+\tilde{a}+\tilde{b}; \zeta), \tag{5.26}$$

$$\zeta^{c-\tilde{a}-\tilde{b}} \cdot {}_2F_1(c-\tilde{b}, 1-\tilde{b}; 1+c-\tilde{a}-\tilde{b}; \zeta) = (1-\zeta)^{-\tilde{a}+\tilde{b}}\zeta^{c-\tilde{a}-\tilde{b}} \cdot {}_2F_1(1-\tilde{a}, c-\tilde{a}; \tilde{A}_0+c; \zeta), \tag{5.27}$$

$$\zeta^{-\tilde{a}} \cdot {}_2F_1(c-\tilde{b}, \tilde{a}; c; \zeta^{-1}) = (1-\zeta^{-1})^{\tilde{b}-\tilde{a}}\zeta^{-\tilde{a}} \cdot {}_2F_1(c-\tilde{a}, \tilde{b}; c; \zeta^{-1}), \tag{5.28}$$

$$\zeta^{-1+c-\tilde{a}} \cdot {}_2F_1(1-\tilde{b}.1-c+\tilde{a}; 2-c; \zeta^{-1}) = (1-\zeta^{-1})^{\tilde{b}-\tilde{a}}\zeta^{-1+c-\tilde{a}} \cdot {}_2F_1(1-\tilde{a}, 1-c+\tilde{b}; 2-c; \zeta^{-1}), \tag{5.29}$$

where $\zeta^{-1} = \frac{1}{1-x} = \frac{t}{t-1}$ and $1-\zeta^{-1} = \frac{1}{1-t}$. By multiplying these equations by $x^{\tilde{\alpha}}$, and then replacing ζ by $1-\frac{1}{t}$ and x by $\frac{1}{t}$, we obtain the remaining eight of Kummer's 24 solutions of Equation (5.16); see Equations 15.5.3-14 in [4], and [15].

5.2.1 24 solutions of Equation (5.1)

Let $u(t)$ be a solution of Equation (5.1) and $\alpha \in \mathbb{C}$. Then Lemma 2.1 shows that $u_\alpha(t) = t^\alpha u(t)$ is a solution of the following equation:

$$[{}_2D_t^0(a-\alpha, b-\alpha) - t \cdot {}_2D_t^0(\tilde{a}-\alpha, \tilde{b}-\alpha)]u_\alpha(t) = 0. \tag{5.30}$$

Comparing this with Equation (5.16), we see that 24 solutionns of Equation (5.30), are obtained from 24 solutions of Equation (5.16), which are given by Equations (5.18)∼(5.19), (5.22)∼(5.23), (5.26)∼(5.29), by replacing $\tilde{a}$ by $\tilde{a}-b$, $\tilde{b}$ by $\tilde{b}-b$, and $c-1$ by $a-b$, and then multiplying t^{-b}.

5.3 Transformation of Equation (5.1) via an exponential function

Theorem 5.2. *Let $u(t)$ be a solution of Equation (5.1), and $\beta_1 \in \mathbb{C}\backslash\{0\}$. Then $v(t) = e^{\beta_1 t}u(t)$ is a solution of*

$$\begin{aligned}[{}_2D_t^0(a,b) - t \cdot {}_2D_t^0(\tilde{a},\tilde{b})]v(t) - 2\beta_1(t-t^2)t\frac{d}{dt}v(t) \\ + [-\beta_1 A_0 t + \beta_1(1+\tilde{a}+\tilde{b}+\beta_1)t^2 - \beta_1^2 t^3]v(t) = 0.\end{aligned} \tag{5.31}$$

Proof. In this case, $u(t)$ satisfies

$$e^{\beta_1 t}[{}_2D_t^0(a,b) - t \cdot {}_2D_t^0(\tilde{a},\tilde{b})]u(t) = 0. \tag{5.32}$$

By using Equation (3.34) for $n = 1$ in this equation, we obtain (5.31). □

The solutions of Equation (5.31), which correspond to solutions of Equation (5.1), given in Lemma 5.3, are obtained from the latters by multiplying by $e^{\beta_1 t}$.

5.4 Transformations of Equations (5.1) and (5.2) via an exponential function

Since Equation (5.2) is obtained from Equation (5.1) by replacing t by x, $u(t)$ by $\tilde{u}(x)$, a by $-\tilde{a}$, b by $-\tilde{b}$, $\tilde{a}$ by $-a$, $\tilde{b}$ by $-b$, we obtain the following theorem by using Theorem 5.2.

Theorem 5.3. *Let $x = \frac{1}{t}$, $u(t)$ and $\tilde{u}(x)$ be solutions of Equations (5.1) and (5.2), respectively, and $\beta_1 \in \mathbb{C}\backslash\{0\}$. Then $\tilde{w}(x) = e^{\beta_1 x}\tilde{u}(x)$ is a solution of*

$$\begin{aligned}[{}_2\tilde{D}_x^0(-\tilde{a},-\tilde{b}) - x \cdot {}_2\tilde{D}_x^0(-a,-b)]\tilde{w}(x) - 2\beta_1(x-x^2)x\frac{d}{dx}\tilde{w}(x) \\ + [-\beta_1\tilde{A}_0 x + \beta_1(1-a-b+\beta_1)x^2 - \beta_1^2 x^3]\tilde{w}(x) = 0,\end{aligned} \tag{5.33}$$

and $w(t) = e^{\beta_1/t}u(t)$ is a solution of

$$\begin{aligned}[{}_2\tilde{D}_t^0(\tilde{a},\tilde{b}) - \frac{1}{t} \cdot {}_2\tilde{D}_t^0(a,b)]w(t) + 2\beta_1(\frac{1}{t} - \frac{1}{t^2})t\frac{d}{dt}w(t) \\ + [-\beta_1\tilde{A}_0\frac{1}{t} + \beta_1(1-a-b+\beta_1)\frac{1}{t^2} - \beta_1^2\frac{1}{t^3}]w(t) = 0.\end{aligned} \tag{5.34}$$

Proof. Equation (5.34) is obtained from Equation (5.33), by using Lemma 1.1. □

Lemma 5.5. *Theorem 5.3 and Lemma 5.4 show that the functions given in Equations (5.15) and (5.14), multiplied by $e^{\beta_1 x}$ or $e^{\beta_1/t}$, are solutions of Equation (5.34) or (5.33).*

6 Conclusion

In [1, 2], the concept of block of classified terms is introduced to a linear differential equation with polynomial coefficients, and solutions of the forms of Equations (1.26) and (1.27) near the origin and infinity, respectively, are presented for Equations (1.8) and (1.10), which consist of two blocks of classified terms. The solutions are summarized in Section 2.

In Section 3∼5, Equations (1.9), (1.10) and (1.8) are expressed by Equations (3.1), (4.1) and (4.2), and (5.1) and (5.2), in this order, and solutions of the forms of Equation (1.26) or (1.27), multiplied by $e^{\beta_1 t} = e^{\beta_1/x}$ or $e^{\beta_1/t} = e^{\beta_1 x}$ or $(1-t)^{\beta}$ or $x^{-\beta}(1-x)^{\beta}$, are given for Equations (3.4), (4.4) and (5.6), which are obtained by a transformation of Equations (3.1), (4.1) and (5.1), via $e^{\beta_1 t}$ or $e^{\beta_1/t}$ or $(1-t)^{\beta}$. These transformed equations consist of three blocks of classified terms.

In [12], transformations of a differential equation without change of singularities are discussed as in Remark 1.11. We note that the differential equations obtained by transformations in Section 3, except in Sections 3.6∼3.8, are regarded as differential equations in a group. Those in Sections 4 and 5, except in Sections 5.3 and 5.4, are also regarded as those in a group.

In Sections 3.6, and 5.3 and 5.4, differential equations are obtained by a transformation from Equations (1.9) and (1.8), respectively. They have different singularities from their respective original equations. One in Section 3.6 is an example of the differential equations, the asymptotic behavior of which was discussed by Stewart [14].

Competing Interests

The author declares no conflict of interests.

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