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Estimation of Reliability under Conditional Stress – Strength Setup based on Weibull Distribution

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The Weibull distribution has been extensively studied and applied across various fields due to its versatility in modeling a wide range of phenomena, especially in reliability engineering, survival analysis, and lifetime modeling. The concept of $R^{|a,b}$, which represents a system's reliability in a conditional stress-strength setup, was proposed by Sabre and Khorshidian (2021). In this research, the problem of estimating reliability of the component is considered when strength variable X and stress variable Y follow independent Weibull distributions with common shapes and different scale parameters under conditional stress-strength setup. The maximum likelihood estimator, asymptotic confidence interval, Bootstrap estimators, Boot-p estimators, and Bayes estimator under-squared error loss function with associated highest posterior density interval are constructed for conditional stress-strength reliability. Simulation study is conducted to estimate mean square error (MSE) of estimator of conditional stress-strength reliability. The real data analysis is also carried out.

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1 Introduction

The capacity of a system or component to carry out a necessary task in a particular environment for a predetermined amount of time is known as reliability. In other words, reliability is the probability that the system will perform satisfactorily for intended period of time. The well-known stress - strength reliability model compares the strength and stress on a certain system. The stress - strength reliability is defined as $R = P(X > Y)$ where X denotes strength of a component and Y denotes the stress. In the stress-strength modeling, R is a measure of component reliability when it is subjected to random stress Y and has strength X. Therefore R and $(1 - R)$ indicate the system performance and probability of system failure respectively. For example, in the event that Y denotes a treatment group and X represents the response for the control group, R represents the treatment's impact.It is not necessary for Stress and Strength to be associated in any way because of their nature. Consequently, a number of authors used the premise that X and Y are independent variables to draw conclusions regarding $P(X < Y)$. Applications of reliability can be found in many fields, including engineering, biostatistics, quality control, economics, psychology, and medicine.

It is said that Church and Harris(1970) [1] first used the term "stress-strength." There are several published studies that looked at different alternatives for distributions of stress and strength. The initial work by Owen et al (1964) [2] focused on constructing confidence limits for the probability $P(X < Y)$ assuming dependence or independence between normally distributed random variables X and Y. Subsequent research expanded the estimation of this probability for various distributions, including exponential, normal, Pareto, and even broader exponential families. For instance, Kelly et al (1976) [3], Tong (1974) [4], Church and Harris (1970) [1], Beg and Singh (1979) [5], and Tong (1977) [6] contributed to estimating $P(X < Y)$ under different distributional assumptions. The introduction of time-dependent models by Bilikam (1985) [7] was a significant step. Bilikam's model considered stress and strength as continuous random processes. In this framework, X and Y were assumed to be stochastically independent but related through time-dependent parameters $\theta_1(t)$ and $\theta_2(t)$. Erylmaz S (2011) [8] investigated stress-strength reliability within the framework of multi-state system modeling. Recently, Pandit and Joshi (2018) [9] studied stress - strength reliability for generalized Pareto distribution.

In usual situation it is known that X and Y are bigger than two fixed values. Especially, when X and Y are two components of the system, and these two components have been worked till a known time, and need to draw inference on R. For purposes of illustration, a vehicle with an engine and brakes is taken into consideration. The engine is thought of as a strength component in the vehicle, and the brake is thought of as a stress component. The engine's lifetime is represented by X, while the brake's is represented by Y. The vehicle will run if the engine is more powerful than the brake, which means the vehicle will run if and only if $P(X > Y)$ is true. Since it is assumed that the vehicle has been driven for an hour, $P(X > Y|X > 1, Y > 1)$ is used to evaluate how reliable the vehicle's performance is. Furthermore, in the circumstance when the engine has been turned on for 30 minutes before driving, the suitable measure of reliability is $P(X > Y|X > 1.5, Y > 1)$. (Refer Saber and Khorshidian(2021)). Motivated by this, Saber and Khorshidian(2021) [10] introduced the conditional stress - strength reliability $R^{|a,b}$.

Later researchers such as Raid et al (2021) [11], and Sabre et al (2021) [12] have attempted to estimate reliability under a conditional stress-strength setup in recent years. They evaluated the system's reliability in a conditional stress-strength setup where the stress-strength variable follows Kumaraswamy, and Generalised exponential lifetime distributions. In all these studies the authors considered the estimation of reliability using classical and Bayesian approaches. They also considered bootstrap confidence intervals. In a conditional stress-strength setup, a system's reliability is represented by $R^{|a,b}$ was introduced by Saber and Khorshidian (2021). The reliability under conditional stress-strength reliability is given by:

$$R^{|a,b} = P(X > Y|X > a, Y > b) = \begin{cases} \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(b)\bar{F}_Y(b)} & a = b \\ \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(a)\bar{F}_Y(b)} & a < b \\ \frac{\int_a^\infty F_Y(x) f_X(x) dx - F_Y(b)\bar{F}_X(a)}{\bar{F}_X(a)\bar{F}_Y(b)} & a > b \end{cases} \tag{1.1}$$

The Weibull distribution is widely researched and utilized in different domains for its capacity to predict a board spectrum of occurrences, particularly in reliability engineering, survival analysis, and lifetime modeling. Its high flexibility and wide range of shapes allow it to have failure rates that are increasing constant, and decreasing. As a result, it has a wide range of uses, including in hydrology, industrial engineering, weather forecasting, and insurance. The Weibull distribution with parameter (γ, θ) is denoted by $W(\gamma, \theta)$. The cumulative distribution function (cdf) and the probability density function (pdf) of this distribution are, respectively,

$$F(z) = 1 - e^{(-\gamma z^\theta)}, z > 0, \gamma, \theta > 0 \tag{1.2}$$

And

$$f(z) = \gamma\theta z^{\theta-1} e^{(-\gamma z^\theta)}, z > 0, \gamma, \theta > 0 \tag{1.3}$$

Nelson (1982) [13] used the Weibull distribution in reliability engineering. Krishnamoorthy et. al (2010) [14] constructed confidence interval for reliability involving Weibull models. Kundu and Gupta (2006) [15] studied the estimate of $R = P(Y < X)$ where $X \sim W(\alpha, \theta_1)$ and $X \sim W(\alpha, \theta_2)$ are two independent Weibull distributions with distinct scale parameters but the same shape parameter. But this paper studies the conditional stress–strength model for Weibull distribution.

The remainder of this paper is structured as follows: Section 2 discusses how $R^{|a,b}$ is derived in the case of the Weibull distribution. Section 3 presents the Maximum Likelihood estimator (MLE) of reliability, $R^{|a,b}$, along with its associated asymptotic distribution and confidence interval. Section 4 considers the bootstrap method used to estimate $R^{|a,b}$ and construct dependability confidence intervals. The Bayesian technique to reliability estimate is discussed in Section 5. Sections 6 and 7 present the simulation findings and real data analysis, respectively.

2 Conditional Stress – Strength Reliability for Weibull Distribution

When the variable strength (X) and stress (Y) are independently distributed with $W(\gamma_1, \theta)$ and $W(\gamma_2, \theta)$, respectively, the conditional reliability of the stress-strength model is derived in this section. Below is a result representing the reliability in a conditional stress-strength set-up.

Result: Let X and Y be independent random variables from the Weibull distributions with parameters (γ_1, θ) and (γ_2, θ) respectively, that is $X \sim W(\gamma_1, \theta)$ and $Y \sim W(\gamma_2, \theta)$. Based on conditional stress - strength model, the reliability $R^{|a,b}$ is given by:

$$R^{|a,b} = \begin{cases} \frac{\gamma_2}{\gamma_1 + \gamma_2} & a = b \\ \frac{\gamma_2}{\gamma_1 + \gamma_2} e^{-\gamma_1 (b^\theta - a^\theta)} & a < b \\ 1 - \frac{\gamma_1}{\gamma_1 + \gamma_2} e^{-\gamma_2 (a^\theta - b^\theta)} & a > b \end{cases} \tag{2.1}$$

Proof: Suppose the strength variable X follows $W(\gamma_1, \theta)$ and stress variable Y follows $W(\gamma_2, \theta)$ distributions.

Case $a = b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1.1), (1.2), (1.3) as

$$R^{|a,b} = \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(b)\bar{F}_Y(b)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} \tag{2.2}$$

Case $a < b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1.1), (1.2), (1.3) as

$$R^{|a,b} = \frac{\bar{F}_Y(b) - \int_b^\infty F_X(y) f_Y(y) dy}{\bar{F}_X(a)\bar{F}_Y(b)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} e^{-\gamma_1 (b^\theta - a^\theta)} \tag{2.3}$$

Case $a > b$: The reliability under conditional stress - strength set up, $R^{|a,b}$ can be derived from (1.1), (1.2), (1.3) as

$$R^{|a,b} = \frac{\int_a^\infty F_Y(x) f_X(x) dx - F_Y(b)\bar{F}_X(a)}{\bar{F}_X(a)\bar{F}_Y(b)} = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{\lambda_1 (e^{a^\theta} - e^{b^\theta})} = 1 - \frac{\gamma_1}{\gamma_1 + \gamma_2} e^{-\gamma_2 (a^\theta - b^\theta)} \tag{2.4}$$

Combining (2.2), (2.3), and (2.4) together we get (2.1).

3 Likelihood Inference of Conditional Reliability

This section deduces the maximum likelihood estimator (MLE) of $R^{|a,b}$ and establishes the asymptotic distribution of the MLE of $R^{|a,b}$. The confidence intervals are constructed using the asymptotic distribution of MLE of $R^{|a,b}$. Suppose that two random samples, $(X_1, X_2, ...X_n)$ and $(Y_1, Y_2, ...Y_m)$, of size n and m, respectively, from $W(\gamma_1, \theta)$ and $W(\gamma_2, \theta)$. The likelihood function for X and Y respectively given in (3.1) and (3.2).

$$L(\gamma_1, \theta|\underline{x}) = \gamma_1^n \theta^n \left(\prod_{i=1}^n x_i^{\theta-1} \right) e^{\sum_{i=1}^n -\gamma_1 x_i^\theta} \tag{3.1}$$

$$L(\gamma_2, \theta|\underline{y}) = \gamma_2^m \theta^m \left(\prod_{i=1}^m y_i^{\theta-1} \right) e^{\sum_{i=1}^m -\gamma_2 y_i^\theta} \tag{3.2}$$

Then, the joint log-likelihood function is given by

$$\begin{aligned} l(\gamma_1, \gamma_2, \theta|\underline{x}, \underline{y}) = \; & n \log \gamma_1 + m \log \gamma_2 + (n+m) \log \theta - \gamma_1 \sum_{i=1}^n x_i^\theta \\ & - \gamma_2 \sum_{i=1}^m y_i^\theta + (\theta - 1) \left(\sum_{i=1}^n \log x_i + \sum_{i=1}^m \log y_i \right) \end{aligned} \tag{3.3}$$

The likelihood equations are obtained as

$$\frac{\partial l}{\partial \gamma_1} = \frac{n}{\gamma_1} - \sum_{i=1}^n x_i^\theta = 0 \tag{3.4}$$

$$\frac{\partial l}{\partial \gamma_2} = \frac{m}{\gamma_2} - \sum_{i=1}^m y_i^\theta = 0 \tag{3.5}$$

$$\frac{\partial l}{\partial \theta} = \frac{m+n}{\theta} + \sum_{i=1}^n \log x_i + \sum_{i=1}^m \log y_i - \gamma_1 \sum_{i=1}^n x_i^\theta \log x_i - \gamma_2 \sum_{i=1}^m y_i^\theta \log y_i = 0 \tag{3.6}$$

The MLE of the parameters γ_1 and γ_2 are given by

$$\hat{\gamma_1} = \frac{n}{\sum_{i=1}^{n} x_i^{\hat{\theta}}} \tag{3.7}$$

$$\hat{\gamma_2} = \frac{m}{\sum_{i=1}^{m} y_i^{\hat{\theta}}} \tag{3.8}$$

And $\hat{\theta}$ is the MLE of the parameter θ which is obtained by solving non-linear equation (3.9)

$$h(\theta) = (n + m)\left\{\hat{\gamma_1}\sum_{i=1}^{n} x_i^{\theta}\log x_i + \hat{\gamma_2}\sum_{i=1}^{m} y_i^{\theta}\log y_i - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{m}\log y_i\right\}^{-1} \tag{3.9}$$

The solution $\hat{\theta}$ to a nonlinear equation (3.9) can be found through an iterative process. This process continues until the difference between consecutive values of θ_j and θ_{j+1} becomes sufficiently small that is $|\theta_j - \theta_{j+1}|$ is very small. This indicates that the iterations have reached a point where terminating them is appropriate, as they have likely converged to the solution. The MLE of $R^{|a,b}$ is obtained as

$$\hat{R^{|a,b}} = \begin{cases} \frac{\hat{\gamma_2}}{\hat{\gamma_1}+\hat{\gamma_2}} & a = b \\ \frac{\hat{\gamma_2}}{\hat{\gamma_1}+\hat{\gamma_2}} e^{-\hat{\gamma_1}(b^{\hat{\theta}} - a^{\hat{\theta}})} & a < b \\ 1 - \frac{\hat{\gamma_1}}{\hat{\gamma_1}+\hat{\gamma_2}} e^{-\hat{\gamma_2}(a^{\hat{\theta}} - b^{\hat{\theta}})} & a > b \end{cases} \tag{3.10}$$

Asymptotic Confidence Interval

The asymptotic distributions of $\hat{\beta} = \left(\hat{\gamma_1}, \hat{\gamma_2}, \hat{\theta}\right)$ and $R^{|a,b}$ are determined. The Fisher information matrix of $\beta = (\gamma_1, \gamma_2, \theta)$, denoted as $I(\beta) = E(JI(\beta))$, where $J(\beta) = [J_{i,j}]_{i,j=1,2,3}$ represents the observed information matrix. The information matrix $J(\beta)$ is given by:

$$J(\beta) = -\begin{pmatrix} \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1^2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1\partial\gamma_2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_1\partial\theta} \\ \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2\partial\gamma_1} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2^2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\gamma_2\partial\theta} \\ \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta\partial\gamma_1} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta\partial\gamma_2} & \frac{\partial^2 l(\gamma_1,\gamma_2,\theta)}{\partial\theta^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} \tag{3.11}$$

and the elements of $J(\beta)$ are as follows:

$$J_{11} = -\frac{n}{\gamma_1^2} \qquad J_{22} = -\frac{m}{\gamma_2^2} \qquad J_{12} = J_{21} = 0;$$

$$J_{13} = J_{31} = -\sum_{i=1}^{n} x_i^{\theta}\log x_i;$$

$$J_{23} = J_{32} = -\sum_{i=1}^{m} y_i^{\theta}\log y_i;$$

$$J_{33} = -\frac{n+m}{\theta^2} - \gamma_1\sum_{i=1}^{n} x_i^{\theta}(\log x_i)^2 - \gamma_2\sum_{i=1}^{m} y_i^{\theta}(\log y_i)^2$$

The components of the Fisher information matrix are derived by computing the expected values of the observed matrix $J(\beta)$, expressed as $I(\beta) = E[J(\beta)]$. This Fisher information matrix $I(\beta)$ can be computed as

$$I(\beta) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{3.12}$$

Where

$$
\begin{aligned}
&I_{11} = -E(J_{11}) = \tfrac{n}{\gamma_1^2} \\
&I_{12} = I_{21} = -E(J_{12}) = -E(J_{21}) = 0 \\
&I_{13} = J_{31} = -E(J_{13}) = -E(J_{31}) = E(\textstyle\sum_{i=1}^{n} x_i^{\theta} \log x_i) \\
&I_{22} = -E(I_{22}) = \tfrac{m}{\gamma_2^2} \\
&I_{23} = I_{32} = -E(J_{23}) = -E(J_{32}) = E(\textstyle\sum_{i=1}^{m} y_i^{\theta} \log y_i) \\
&I_{33} = -E(J_{33}) = \tfrac{n+m}{\theta^2} + \gamma_1 E\left(\textstyle\sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2\right) + \gamma_2 E\left(\textstyle\sum_{i=1}^{m} y_i^{\theta} (\log y_i)^2\right)
\end{aligned}
$$

As $n \to \infty$ and $m \to \infty$, then by using of multivariate central limit theorem (CLT) of $\hat{\theta}$, we have $\hat{\theta} \to N_3(\theta, \Sigma)$ where $\hat{\theta} = \left(\hat{\lambda_1}, \hat{\lambda_2}, \hat{\beta}\right)$ and Σ is inverse of the Fisher information matrix:

$$
\Sigma = \frac{1}{\det I(\beta)} \begin{pmatrix} I_{22}I_{33} - I_{32}I_{23} & I_{13}I_{32} - I_{12}I_{33} & I_{12}I_{23} - I_{13}I_{22} \\ I_{23}I_{31} - I_{21}I_{33} & I_{11}I_{33} - I_{13}I_{31} & I_{13}I_{21} - I_{11}I_{23} \\ I_{21}I_{32} - I_{22}I_{31} & I_{12}I_{31} - I_{11}I_{32} & I_{11}I_{22} - I_{12}I_{21} \end{pmatrix} \tag{3.13}
$$

The asymptotic distribution of $R^{|a,b}$ is derived by employing the multivariate Delta approach as outlined in the following lemma.

Lemma: Consider a sequence $\{X_n\}_{n=1}^{\infty}$ of random vectors converging in distribution to $N_k(\mu, \Sigma)$ i.e. $X_n \to N_k(\mu, \Sigma)$. Let $g(x) : R_k \to R$ be a function continuous in its first partial derivatives, and let $\sigma^2 = \Delta^T \Sigma \Delta > 0$, where $\Delta = \frac{\partial g(\mu)}{\partial \mu}$. Then, $\frac{g(X_n) - g(\mu)}{\sigma} \to N(0, 1)$

Result: As sample size increases, $\frac{\hat{R}^{|a,b} - R^{a,b}}{\sigma}$ converges in distribution to a standard normal distribution. Mathematically, As $n \to \infty$ and $m \to \infty$, then

$$
\frac{\hat{R}^{|a,b} - R^{a,b}}{\sigma} \xrightarrow{d} N(0, 1) \tag{3.14}
$$

Here, σ^2 is obtained with the help of $g(\beta)\Sigma g(\beta)^T$ for different cases $a = b, a < b, a > b$.
Also $\beta = (\lambda_1, \lambda_2, \beta)$, and $g(\beta)$ denotes the derivative of $R^{|a,b}$ with respect to β. It's represented as $g(\beta) = \left(\frac{\partial R^{|a,b}}{\partial \gamma_1}, \frac{\partial R^{|a,b}}{\partial \gamma_2}, \frac{\partial R^{|a,b}}{\partial \theta}\right)$. Additionally, Σ stands for the inverse of the Fisher information matrix.

Proof: Take the partial derivatives of $R^{|a,b}$, as outlined in equation (2.1), with respect to γ_1, γ_2, and θ to derive equations (22), (23), and (24) correspondingly.

$$
\frac{\partial R^{|a,b}}{\partial \gamma_1} = \begin{cases} -\frac{\gamma_2}{(\gamma_1+\gamma_2)^2} & a = b \\ \frac{\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} \left(a^\theta - b^\theta - \frac{1}{(\gamma_1+\gamma_2)}\right) & a < b \\ -\frac{\gamma_2}{(\gamma_1+\gamma_2)^2} e^{-\gamma_2(a^\theta - b^\theta)} & a > b \end{cases} \tag{3.15}
$$

$$
\frac{\partial R^{|a,b}}{\partial \gamma_2} = \begin{cases} \frac{\gamma_1}{(\gamma_1+\gamma_2)^2} & a = b \\ \frac{\gamma_1}{(\gamma_1+\gamma_2)^2} e^{-\gamma_1(a^\theta - b^\theta)} & a < b \\ \frac{\gamma_1}{\gamma_1+\gamma_2} e^{-\gamma_2(b^\theta - a^\theta)} \left(a^\theta - b^\theta + \frac{1}{\gamma_1+\gamma_2}\right) & a > b \end{cases} \tag{3.16}
$$

$$
\frac{\partial R^{|a,b}}{\partial \theta} = \begin{cases} 0 & a = b \\ \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_1(b^\theta - a^\theta)} \left(a^\theta \log a - b^\theta \log b\right) & a < b \\ \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{-\gamma_2(a^\theta - b^\theta)} \left(a^\theta \log a - b^\theta \log b\right) & a > b \end{cases} \tag{3.17}
$$

Since $(\hat{\beta}-\beta) \rightarrow N(\beta, \Sigma)$, Using Cramer's theorem,

$$\left(g(\hat{\beta})-g(\beta)\right) \rightarrow N(0, g(\beta)\Sigma g(\beta)^T) \tag{3.18}$$

Where $g(\beta)=\frac{\partial R^{|a,b}}{\partial \beta}=\left(\frac{\partial R^{|a,b}}{\partial \gamma_1}, \frac{\partial R^{|a,b}}{\partial \gamma_2}, \frac{\partial R^{|a,b}}{\partial \theta}\right)$

The confidence interval at the $(1-\alpha)\%$ level for $R^{|a,b}$ is expressed as

$$\begin{aligned} R^{|a,b} &\in \left(\hat{R^{|a,b}}-Z_{1-\frac{\alpha}{2}}\sigma_1^2, \hat{R^{|a,b}}+Z_{1-\frac{\alpha}{2}}\sigma_1^2\right) \qquad a=b \\ R^{|a,b} &\in \left(\hat{R^{|a,b}}-Z_{1-\frac{\alpha}{2}}\sigma_2^2, \hat{R^{|a,b}}+Z_{1-\frac{\alpha}{2}}\sigma_2^2\right) \qquad a<b \\ R^{|a,b} &\in \left(\hat{R^{|a,b}}-Z_{1-\frac{\alpha}{2}}\sigma_3^2, \hat{R^{|a,b}}+Z_{1-\frac{\alpha}{2}}\sigma_3^2\right) \qquad a>b \end{aligned} \tag{3.19}$$

The values of $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are determined by evaluating $g(\beta)\Sigma g(\beta)^T$ under various conditions namely when $a=b$, when $a<b$, and when $a>b$.

4 Bootstrap Approach for $R^{|a,b}$

This section examines the confidence interval for $R^{|a,b}$ using the parametric bootstrap method. The process of generating parametric bootstrap samples for $R^{|a,b}$, as suggested by Efron and Tibshirani (1993) [16], is detailed here:

- Calculate $\hat{\gamma_1}, \hat{\gamma_2}, \hat{\theta}, \hat{R^{|a,b}}$, which represent the maximum likelihood estimators (MLEs) of $\gamma_1, \gamma_2, \theta, R^{|a,b}$, respectively, using the samples $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ...Y_m)$.
- Generate independent bootstrap samples $X^* = (X_1^{;}X_2^{;}..., X_n^{)}$ drawn from $W(\gamma_1, \theta)$ and $Y^{=}(Y_1^{;}Y_2^{;}...Y_m^{)}$ drawn from $W(\gamma_2, \theta)$. Using the bootstrap data, calculate bootstrap estimations for the parameters, denoted as $\gamma_1^*, \gamma_2^*, \theta^*$, and $R^{*|a,b}$.
- Iterate the previously mentioned process B times to generate a series of bootstrap samples for $R^{|a,b}$, denoted as $R_1^{*|a,b}, R_2^{*|a,b}, ...R_B^{*|a,b}$.

With the obtained bootstrap samples of R, the $100(1-\alpha\%)$ percentile bootstrap confidence interval for $R^{|a,b}$ is constructed and presented as follows:

$$\left(\hat{R}^{*|a,b}_{\left(\frac{\alpha}{2}\right)}, \quad \hat{R}^{*|a,b}_{\left(1-\frac{\alpha}{2}\right)}\right) \tag{4.1}$$

Where $\hat{R}^{*|a,b}_{(\alpha)}$ is the quantile of order γ .

5 Bayesian Inference on Conditional Reliability

This section related to Bayes estimation of reliability within a conditional stress-strength framework. For Bayesian studies one needs to select prior for the parameters of the distribution under study. Here the conjugate prior is selected for the parameters of the distribution.A conjugate prior is a specific prior that results in a posterior that shares the same distribution as the prior. The prior distribution for the parameters taken as Gamma prior because of its conjugate nature. The prior distributions for the parameters γ_1, γ_2, and θ are taken as $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, and $Gamma(a_3, b_3)$, respectively. It is assumed that all prior distributions are independent. The joint prior density is given by:

$$\pi\left(\gamma_1, \gamma_2, \theta\right)=\frac{b_1^{a_1}\gamma_1^{a_1-1}e^{-b_1\gamma_1}}{\Gamma\left(a_1\right)}\frac{b_2^{a_2}\gamma_2^{a_2-1}e^{-b_2\gamma_2}}{\Gamma\left(a_2\right)}\frac{b_3^{a_3}\gamma_3^{a_3-1}e^{-b_3\gamma_3}}{\Gamma\left(a_3\right)} \tag{5.1}$$

Hence, under the assumption of independence among γ_1, γ_2, and θ, the joint posterior density of γ_1, γ_2, and θ is expressed as:

$$\pi\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right)=\frac{L\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right)\pi(\gamma_1)\pi(\gamma_2)\pi(\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\gamma_1, \gamma_2\theta|\underline{x}, \underline{y}\right)\pi(\gamma_1)\pi(\gamma_2)\pi(\theta)d\theta d\gamma_1 d\gamma_2} \tag{5.2}$$

Now (5.2) can be obtained using (3.1), (3.2) and (5.1)

$$\pi\left(\gamma_1, \gamma_2, \theta | \underline{x}, \underline{y}\right) = \frac{\varrho}{\int_0^\infty \int_0^\infty \int_0^\infty \varrho d\theta d\gamma_1 d\gamma_2} \tag{5.3}$$

$$\varrho = \frac{\gamma_1^{n+a_1-1} b_1^{a_1} e^{-\gamma_1 (b_1 + \sum_{i=1}^n x_i^\theta)}}{\Gamma(a_1)} \frac{\gamma_2^{m+a_2-1} b_2^{a_2} e^{-\gamma_2 (b_2 + \sum_{i=1}^m y_i^\theta)}}{\Gamma(a_2)} \frac{\theta^{n+m+a_3-1} b_3^{a_3} e^{-\theta b_3}}{\Gamma(a_3)} \prod_{i=1}^n x_i^{\theta-1} \prod_{i=1}^m y_i^{\theta-1} \tag{5.4}$$

The structure of the posterior density doesn't yield a direct explicit Bayes estimator for the model parameters. Thus, the conditional distribution of γ_1, γ_2, and θ can be obtained using the Gibbs sampling technique as follows:

$$\left(\gamma_1 | \gamma_2, \theta, \underline{x}, \underline{y}\right) \sim Gamma\left(n + a_1, b_1 + \sum_{i=1}^n x_i^\theta\right) \tag{5.5}$$

$$\left(\gamma_2 | \gamma_1, \theta, \underline{x}, \underline{y}\right) \sim Gamma\left(m + a_2, b_2 + \sum_{i=1}^m y_i^\theta\right) \tag{5.6}$$

$$\pi\left(\theta | \gamma_1, \gamma_2, \underline{x}, \underline{y}\right) \propto \theta^{n+m+a_3-1} e^{-b_3\theta} \prod_{i=1}^n x_i^{\theta-1} \prod_{i=1}^m y_i^{\theta-1} \tag{5.7}$$

Use gamma distributions to generate random numbers for γ_1 and γ_2. Then, utilizing the "Metropolis-Hastings method," generate random values for θ with a distribution proportional to $N(\theta^{t-1}, k_\theta V_\theta)$, where θ^{t-1} represents the current value of θ, k_θ is the scaling factor, and V_θ is the variance-covariance matrix. Thus, Gibbs sampling is as follows:

1. Set t = 1 and begin with an initial guess of $\theta^{(0)} = \hat{\theta}$.
2. From $Gamma\left(n + a_1, b_1 + \sum_{i=1}^n x_i^\theta\right)$ distribution, generate a random value $\gamma_1^{(t)}$.
3. From $Gamma\left(m + a_2, b_2 + \sum_{i=1}^m y_i^\theta\right)$ distribution, generate a random value $\gamma_2^{(t)}$.
4. Generate $\theta^{(t)}$ from (5.7) $\pi\left(\theta | \gamma_1, \gamma_2, \underline{x}, \underline{y}\right)$ using the "Metropolis - Hastings method" with a proportional distribution as normal distribution.
5. Determine $R^{|a,b(t)}$
6. Assign t to t+1
7. Execute steps 2 to 6 repeatedly for a total of N iterations.

The approximate posterior mean and posterior variance of $R^{|a,b}$ are as follows:

$$\hat{E}\left(R^{|a,b} | \underline{x}, \underline{y}\right) = \frac{1}{N-M} \sum_{t=M+1}^N R^{|a,b(t)} \tag{5.8}$$

and

$$\hat{V}\left(R^{|a,b} | \underline{x}, \underline{y}\right) = \frac{1}{N-M} \sum_{t=M+1}^N \left(R^{|a,b(t)} - \hat{E}\left(R^{|a,b} | \underline{x}, \underline{y}\right)\right)^2 \tag{5.9}$$

Where M is the burn - in period (that is, a number of iterations before the stationary distribution is achieved). Based on the $R^{|a,b(t)}$ value, the Chen and Shao (1999) [17] approach can be utilised to generate a $100(1-\gamma)\%$ HPD credible interval. The HPD credible interval is given as follows:

$$\left(R^{|a,b}_{\left[N\frac{\gamma}{2}\right]}, \qquad R^{|a,b}_{\left[N(1-\frac{\gamma}{2})\right]}\right) \tag{5.10}$$

where $R^{|a,b}_{\left[N\frac{\gamma}{2}\right]}$ and $R^{|a,b}_{\left[N(1-\frac{\gamma}{2})\right]}$ are the $\left[N\frac{\gamma}{2}\right]^{\text{th}}$ smallest integer and the $\left[N(1-\frac{\gamma}{2})\right]^{\text{th}}$ smallest integer of $R^{|a,b(t)}, t = M+1, M+2, ..., N$, respectively.

6 Simulation Study

In this section, a Monte Carlo simulation analysis is carried out to evaluate the performance of the point estimators and confidence intervals constructed in this research. Confidence intervals are compared using length and coverage probability, whereas point estimators are evaluated using mean square error.All predicted values have been rounded to five digits after 5000 iterations. The usage of abbreviations in the tables is as follows: *MLE: Maximum Likelihood Estimation Method, Bootstrap: Bootstrap Estimation, Bayesian: Bayesian Approach, MSE: Mean Square Error, ACI: Asymptotic Confidence interval, BPCI: Boot - percentile Confidence Interval, HPDCI: HPD credible interval* and *CP: Coverage Probability.* (Refer Henningsen et.al (2011) [18] for R coding).

The tables below present the findings of the simulation study.

Table 1. Point Estimation: MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|1,3}$

$\gamma_1 = 0.1, \gamma_2 = 5, \theta = 1.2, R^{|1,3} = 0.74563$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|13}$	MSE	$\hat{R}^{|13}$	MSE	$\hat{R}^{|13}$	MSE
(5, 8)	0.78488	0.01866	0.84987	0.01935	0.78249	0.001358
(8, 5)	0.74916	0.01747	0.62715	0.03286	0.66303	0.006823
(8, 8)	0.932	0.0157	0.60499	0.03728	0.72966	0.000255
(10, 15)	0.83281	0.01318	0.76163	0.00726	0.76702	0.000458
(15, 10)	0.77663	0.01328	0.72186	0.0065	0.66469	0.006552
(15, 15)	0.86497	0.01242	0.8147	0.00773	0.74278	0.000008
(25, 25)	0.84231	0.01152	0.86501	0.01543	0.70308	0.001811
(35, 25)	0.83229	0.01107	0.62369	0.01826	0.67225	0.005385
(25, 35)	0.8672	0.01106	0.74235	0.00295	0.79114	0.002071
(55, 35)	0.88557	0.01084	0.67302	0.00695	0.65436	0.008330
(35, 55)	0.80697	0.01062	0.91114	0.02781	0.81035	0.004188
(55, 55)	0.86641	0.01034	0.81503	0.00568	0.75927	0.000186

Table 2. Interval Estimation: MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|1,3}$

$\gamma_1 = 0.1, \gamma_2 = 5, \theta = 1.2, R^{|1,3} = 0.74563$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.59658, 0.97318)	0.98	(0.64627, 0.97089)	1	(0.5718, 0.92795)	1
(8, 5)	(0.64161, 0.85671)	0.97	(0.35054, 0.87169)	0.96	(0.38581, 0.86223)	1
(8, 8)	(0.6847, 0.9993)	1	(0.32022, 0.83683)	0.99	(0.50684, 0.88681)	1
(10, 15)	(0.68157, 0.88406)	0.99	(0.56917, 0.89642)	1	(0.59379, 0.89036)	0.98
(15, 10)	(0.63924, 0.91402)	1	(0.55164, 0.85493)	1	(0.45146, 0.81974)	1
(15, 15)	(0.65962, 0.93433)	1	(0.69796, 0.91083)	0.98	(0.57578, 0.86352)	1
(25, 25)	(0.68038, 0.90424)	1	(0.78818, 0.92677)	1	(0.54429, 0.81457)	0.99
(35, 25)	(0.72354, 0.92104)	0.99	(0.50469, 0.82535)	1	(0.50736, 0.78325)	1
(25, 35)	(0.69659, 0.93781)	1	(0.61722, 0.83735)	1	(0.66769, 0.87162)	1
(55, 35)	(0.7027, 0.91844)	1	(0.59106, 0.84571)	1	(0.45568, 0.75704)	1
(35, 55)	(0.72903, 0.88492)	1	(0.86778, 0.94599)	1	(0.66245, 0.87915)	1
(55, 55)	(0.7128, 0.92002)	1	(0.75187, 0.86913)	1	(0.57017, 0.83662)	1

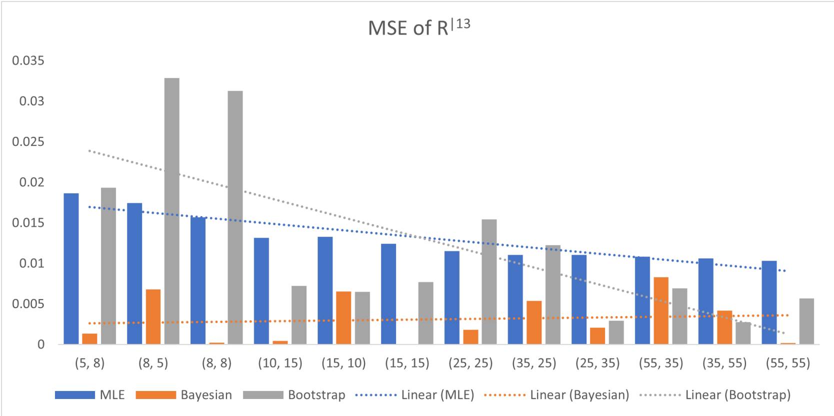

Fig. 1. Plot of mean square error against sample size for $R^{|1,3}$

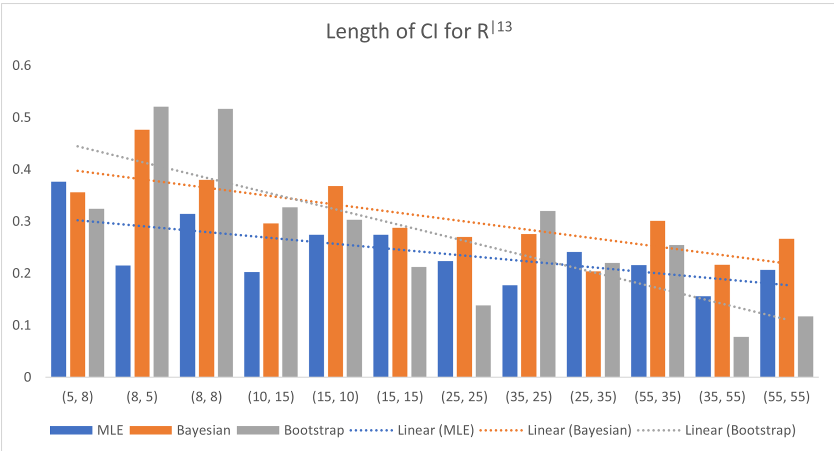

Fig. 2. Plot of length of confidence interval against sample size $R^{|1,3}$

Table 3. Point Estimation: MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|1,1}$

$\gamma_1 = 0.3, \gamma_2 = 2, \theta = 1.2, R^{|1,1} = 0.86957$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|11}$	MSE	$\hat{R}^{|11}$	MSE	$\hat{R}^{|11}$	MSE
(5, 8)	0.8523	0.00695	0.86153	0.00839	0.723	0.00148
(8, 5)	0.87171	0.00639	0.89322	0.00646	0.75364	0.00344
(8, 8)	0.86785	0.0052	0.84782	0.00752	0.75601	0.00289
(10, 15)	0.93748	0.00365	0.88731	0.00313	0.72389	0.00222
(15, 10)	0.89505	0.00318	0.80344	0.00999	0.82335	0.00214
(15, 15)	0.86349	0.00272	0.96697	0.00988	0.75172	0.00389
(25, 25)	0.83223	0.00165	0.85897	0.00198	0.82459	0.00202
(35, 25)	0.91685	0.00123	0.87983	0.00144	0.82519	0.00197
(25, 35)	0.86522	0.00135	0.85666	0.00171	0.75399	0.00336
(55, 35)	0.89775	0.00081	0.88067	0.00099	0.8017	0.00461
(35, 55)	0.89498	0.00089	0.87082	0.00095	0.73925	0.00198
(55, 55)	0.87144	0.00071	0.86173	0.00084	0.83581	0.00114

Table 4. Interval Estimation: MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|1,1}$

$\gamma_1 = 0.3, \gamma_2 = 2, \theta = 1.2, R^{\|1,1} = 0.86957$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.67248, 0.93201)	0.97	(0.6354, 0.9839)	1	(0.48316, 0.89854)	1
(8, 5)	(0.73797, 0.95205)	0.98	(0.71134, 0.99094)	1	(0.65171, 0.98848)	1
(8, 8)	(0.73088, 0.948)	0.96	(0.64733, 0.97376)	1	(0.6671, 0.98207)	1
(10, 15)	(0.68575, 0.99102)	1	(0.76676, 0.97313)	1	(0.55023, 0.96188)	0.98
(15, 10)	(0.64826, 0.94185)	1	(0.64299, 0.93703)	1	(0.62091, 0.93895)	1
(15, 15)	(0.76507, 0.96192)	1	(0.82189, 0.99371)	1	(0.64202, 0.94291)	1
(25, 25)	(0.71168, 0.95277)	1	(0.76833, 0.93231)	1	(0.7613, 0.9882)	0.99
(35, 25)	(0.78455, 0.95916)	1	(0.80533, 0.94524)	1	(0.7287, 0.99056)	1
(25, 35)	(0.77448, 0.95596)	1	(0.77062, 0.92582)	1	(0.6724, 0.97246)	1
(55, 35)	(0.75121, 0.94429)	0.99	(0.81887, 0.93183)	1	(0.64883, 0.90622)	1
(35, 55)	(0.82626, 0.96371)	0.98	(0.80376, 0.92388)	1	(0.52835, 0.88924)	1
(55, 55)	(0.81896, 0.92391)	1	(0.8047, 0.91143)	1	(0.71634, 0.91979)	1

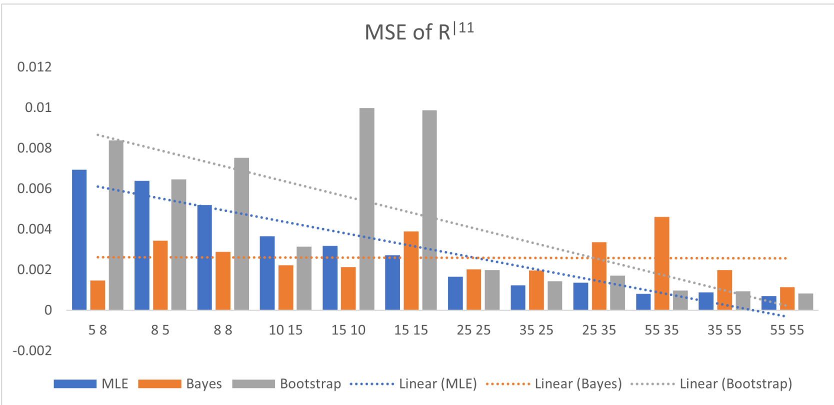

Fig. 3. Plot of mean square error against sample size for $R^{|1,1}$

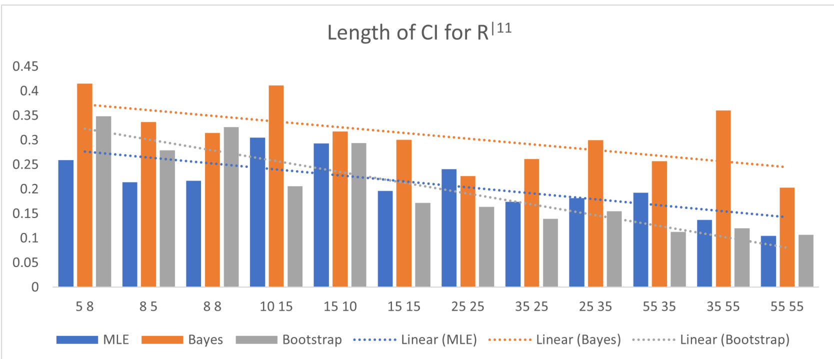

Fig. 4. Plot of length of confidence interval against sample size $R^{|1,1}$

Table 5. Point Estimation: MLE, Bootstrap and Bayesian estimators and corresponding MSE for $R^{|3,1}$

$\gamma_1 = 0.09, \gamma_2 = 0.6, \theta = 1.2, R^{|3,1} = 0.97476$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	$\hat{R}^{|31}$	MSE	$\hat{R}^{|31}$	MSE	$\hat{R}^{|31}$	MSE
(5, 8)	0.9687	0.00343	0.85812	0.00231	0.95885	0.000250
(8, 5)	0.98851	0.00329	0.98167	0.00059	0.83825	0.001860
(8, 8)	0.96421	0.00244	0.95606	0.00211	0.89779	0.001920
(10, 15)	0.96505	0.00179	0.96942	0.00065	0.9518	0.000530
(15, 10)	0.94228	0.00198	0.98704	0.0003	0.9489	0.000670
(15, 15)	0.97674	0.0016	0.95394	0.00123	0.97322	0.000240
(25, 25)	0.97896	0.00134	0.98662	0.00027	0.92629	0.000230
(35, 25)	0.97079	0.00136	0.99086	0.00031	0.98447	0.000940
(25, 35)	0.97354	0.00119	0.98117	0.00017	0.95824	0.000270
(55, 35)	0.87062	0.00118	0.9747	0.0002	0.9537	0.000440
(35, 55)	0.95289	0.00105	0.94802	0.00101	0.99417	0.000380
(55, 55)	0.96175	0.00101	0.96488	0.00025	0.948	0.000720

Table 6. Interval Estimation: MLE, Bootstrap and Bayesian estimators and corresponding CP for $R^{|3,1}$

$\gamma_1 = 0.09, \gamma_2 = 0.6, \theta = 1.2, R^{|3,1} = 0.97476$						
2*(n, m)	**MLE**		**Bootstrap**		**Bayesian**	
	ACI	CP	BPCI	CP	HPDCI	CP
(5, 8)	(0.90464, 0.99999)	0.97	(0.81483, 0.99195)	1	(0.8977, 0.98992)	0.98
(8, 5)	(0.86704, 0.99996)	0.98	(0.91424,1)	1	(0.86695, 0.9926)	1
(8, 8)	(0.91003, 0.99999)	1	(0.85141, 0.99988)	1	(0.86843, 0.99659)	0.96
(10, 15)	(0.92433, 0.99979)	1	(0.9053, 0.99905)	1	(0.89114, 0.98423)	1
(15, 10)	(0.8685, 0.99735)	1	(0.95501, 0.9996)	1	(0.86078, 0.98171)	1
(15, 15)	(0.94874, 0.9974)	0.99	(0.88348, 0.99366)	1	(0.93811, 0.98922)	1
(25, 25)	(0.96476, 0.99316)	1	(0.95689, 0.99937)	1	(0.94116, 0.99994)	1
(35, 25)	(0.89482, 0.99677)	1	(0.9731, 0.99918)	1	(0.94379, 0.99314)	0.99
(25, 35)	(0.95388, 0.9932)	1	(0.95416, 0.99748)	1	(0.92852, 0.98775)	1
(55, 35)	(0.86689, 0.99435)	1	(0.94198, 0.99566)	1	(0.92313, 0.98316)	1
(35, 55)	(0.92614, 0.99765)	1	(0.91049, 0.97787)	1	(0.89903, 0.99878)	1
(55, 55)	(0.93624, 0.98726)	1	(0.9383, 0.98585)	1	(0.91228, 0.98095)	1

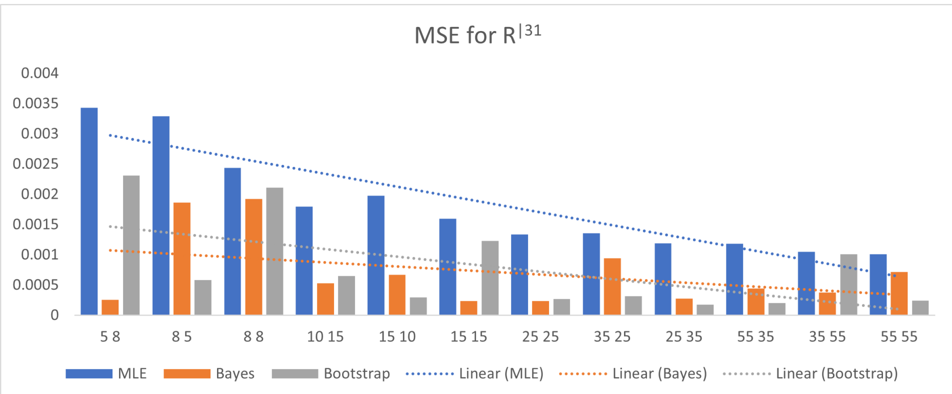

Fig. 5. Plot of mean square error against sample size for $R^{|3,1}$

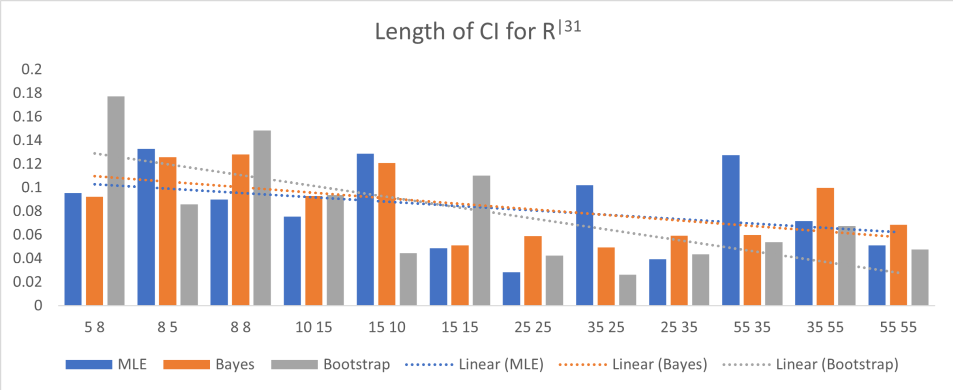

Fig. 6. Plot of length of confidence interval against sample size $R^{|3,1}$

From the simulation study, it is observed that

1. MSE's of MLEs decreased as sample size increases.
2. Bayesian methods often showed lower MSE as compared to MLE and Bootstrap.
3. As sample size increases, length of the confidence interval is shortened.

7 Data Analysis

This section is about real data analysis. Badar and Priest (1982) [19] published the first data on fibre strength (in GPA). The data provide the strength values measured in GPA for single carbon Fibre and impregnated 1000-carbon Fibre tows. Single fibres were tension tested at gauge lengths of 1, 10, 20, and 50mm. Impregnated tows of 1000 Fibre were tested at gauge lengths of 20, 50, 150, and 300. The two data sets presented here are for single fibres tested under tension at gauge lengths of 10 mm (Data I) and 20 mm (Data II), with sample sizes of n = 63 and m = 69, respectively.

Data I: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977 2.996 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377 3.408 3.435 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Data II: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770 2.773 2.800 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585..

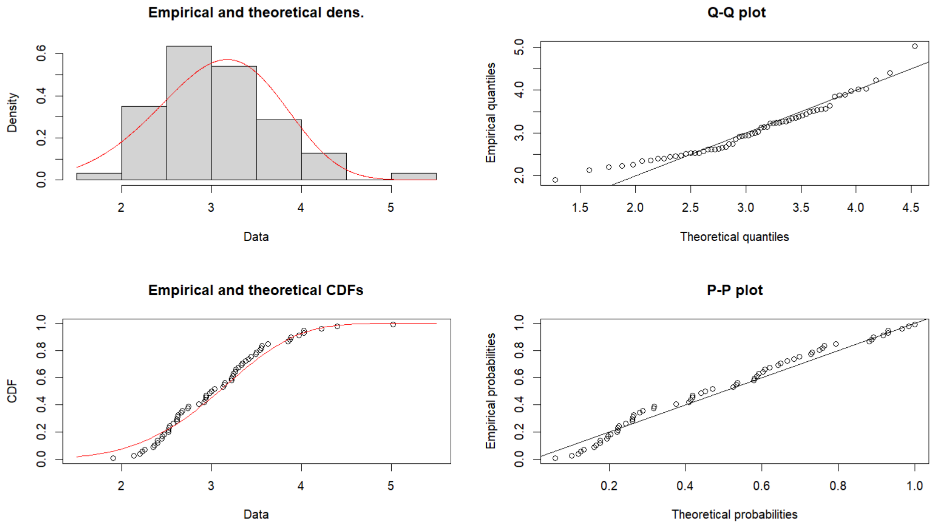

Fig. 7. Plot of goodness of fit for Data I

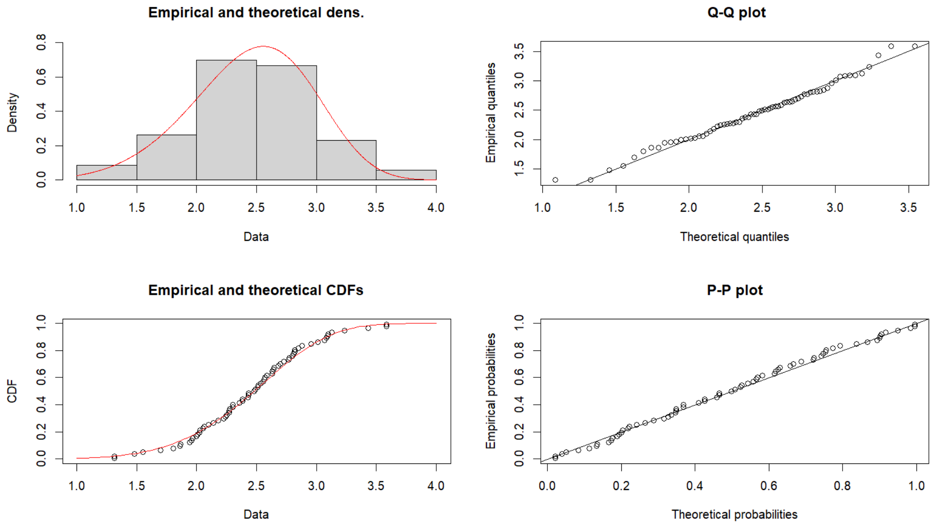

Fig. 8. Plot of goodness of fit for Data II

From the goodness of fit graphs (Figure 7 and Figure 8), it can be observe that Weibull distribution fits the data. Further, tested the model's validity using the Kolmogorov-Smirnov (K-S) test for each data set. The K-S distances (p-value) for data sets I and II were found to be 0.087616 (0.7188) and 0.056128 (0.9816). Based on K-S distance and p-value it can be concluded that the Weibull distribution fits the both data sets. The estimated parameters are $\hat{\gamma}_1 = 0.001787$, $\hat{\gamma}_2 = 0.006032$ and $\hat{\theta} = 5.2619$. The $R^{|a,b}$ estimated under the proposed estimation methods namely, the maximum likelihood estimation, Bootstrap estimation, and Bayes estimation and results are given in table 7. The asymptotic confidence interval (ACI), Boot P confidence interval (BPCI) and HPD credible interval (HPDCI) are also constructed and produced in table 7. It can be observed that the outcomes of the MLE and Bootstrap methods are equivalent in terms of point estimation and that the confidence intervals for Bayes estimation and Bootstrap estimation are similar in terms of length.

Table 7. Results of Data Analysis

Method	$R^{\|a,b}$	Confidence Interval
MLE	$\hat{R}_{mle}^{\|1.9,1.3} = 0.80387$	ACI = (0.74416, 0.86357)
Bootstrap	$\hat{R}_{boot}^{\|1.9,1.3} = 0.80316$	BPCI = (0.73191, 0.86254)
Bayes	$\hat{R}_{Bayes}^{\|1.9,1.3} = 0.74735$	HPDCI = (0.67788, 0.81274)

8 Conclusions

In the field of reliability research, the conditional stress-strength model stands out as an innovative extension of the stress-strength model. This particular investigation aims to estimate the reliability under conditional stress-strength scenarios, assuming the stress and strength parameters follows Weibull distributions with same shape parameters but distinct scale parameters. The study employs maximum likelihood estimation, bootstrap estimation and Bayesian approaches to determine the distribution parameters and the conditional reliability. Confidence intervals for conditional reliability are constructed through the MLE and bootstrap method, while HPD credible intervals are also established using Bayesian approach. To illustrate, a simulation study is conducted, revealing that with an increase in sample size, the lengths of the intervals tend to decrease. Moreover, the Mean Squared Error (MSE) of the Bayesian estimates frequently proves to be lower than that of the Maximum Likelihood Estimator (MLE) as well as bootstrap. It is also observed that, the MSE of the MLE decreases as the sample size grows, demonstrating the consistency of this estimator. A real data analysis is also carried out as an application of the same.

Competing Interests

Authors have declared that no competing interests exist.

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