



## The Hamza Distribution with Statistical Properties and Applications

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### Authors' contributions

This work was carried out in collaboration among all authors. Author AA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AA, MJ, SQUA and RT managed the analysis of the study. Author AA managed the literature searches. All authors read and approved the final manuscript.

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## Abstract

This paper suggested a new two parameter distribution named as Hamza distribution. A detailed description about the properties of a suggested distribution including moments, moment generating function, deviations about mean and median, stochastic orderings, Bonferroni and Lorenz curves, Renyi entropy, order statistics, hazard rate function and mean residual function has been discussed. The behavior of a probability density function (p.d.f) and cumulative distribution function (c.d.f) have been depicted through graphs. The parameters of the distribution are estimated by the known method of maximum likelihood estimation. The performance of the established distribution have been illustrated through applications, by which we conclude that the established distribution provide better fit.

**Keywords:** Moments; moment generating function; reliability measures; mean deviations; maximum likelihood function.

**Mathematics subject classification:** 60E05, 62E15.

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## 1 Introduction

Statistical distribution plays a significant role in bio-medicine, engineering, economics and other areas of science. The exponential and gamma distribution are popular distributions and are considered as life time distribution for analyzing statistical data. Among these distributions, exponential distribution has one parameter and holds several interesting statistical properties as it has memory less and constant hazard rate property. In the literature of statistics, various extensions of these distributions have been made for more flexibility. In (1958), Lindley [1], have proposed one parameter life time distribution having the following probability density function.

$$f(y, \beta) = \frac{\beta^2}{(1 + \beta)}(1 + y)e^{-\beta y} \quad ; y > 0, \beta > 0$$

In recent years researchers have made a lot of work on Lindley distribution and they have proposed one as well as two parameter distributions. for modeling different complex data. An elaborate study on Lindley distribution has been done by Ghitney et al. [2], they have laid out that Lindley distribution provide better results than exponential distribution for the waiting times before the service of the bank customers. They also illustrates that the contours of hazard rate function of the Lindley distribution is an increasing function while as the mean residual life is a decreasing function of the random variable. Zakerzadeh and Dolati [3], Nadarajah et al. [4], they extended Lindley distribution with addition of new parameters and expounded the performance of the extended distribution through data sets. In recent years many authors have made different contributions to modify the Lindley distribution. Merovci [5], has introduced transmuted Lindley distribution and discussed its several properties. Sharma et al. [6], has introduced the inverse of the Lindley distribution and propounded its different properties. Shanker et al. [7] developed a new life time distribution named Ishita distribution which were proved superior then exponential and Lindley distribution. K. K Shukla [8], have suggested the Pranav distribution and studied its different properties. Ahmad et al. [9], has introduced transmuted inverse Lindley distributions and estimates its properties. These distributions have benefits as well as drawbacks over one another to analyze more complex data. In this paper an attempt has been made to establish a new two parameter distribution which is more pliable and provide better results than the existing ones. The probability density function of the newly established two parameter distribution follows.

$$f(y, \alpha, \beta) = \frac{\beta^6}{\alpha\beta^5 + 120} \left( \alpha + \frac{\beta}{6}y^6 \right) e^{-\beta y} \quad ; y > 0, \alpha, \beta > 0 \tag{1.1}$$

The proposed distribution is named as Hamza distribution which is a combination of two distributions, Exponential distribution having scale parameter  $\beta$  and gamma distribution  $g(7, \beta)$ . With combining proportion as  $\frac{\alpha\beta^5}{\alpha\beta^5+120}$

$$f(y, \alpha, \beta) = \pi\phi_1(y, \beta) + (1 - \pi)\phi_2(y, \beta)$$

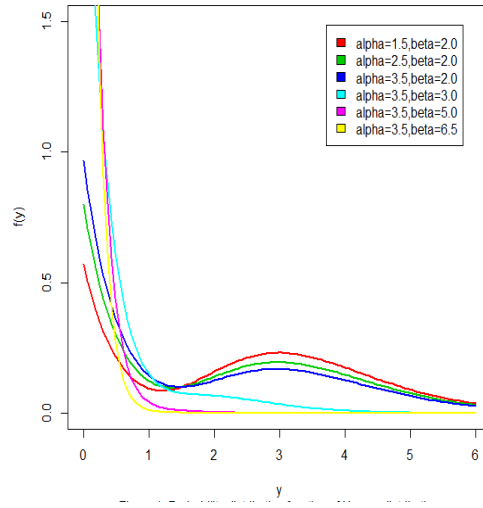
$$\pi = \frac{\alpha\beta^5}{\alpha\beta^5 + 120} \quad , \phi_1(y, \beta) = \beta e^{-\beta y} \quad , \phi_2(y, \beta) = \frac{\beta^7}{720} y^6 e^{-\beta y}$$

The cumulative distribution function of (1.1) is given by

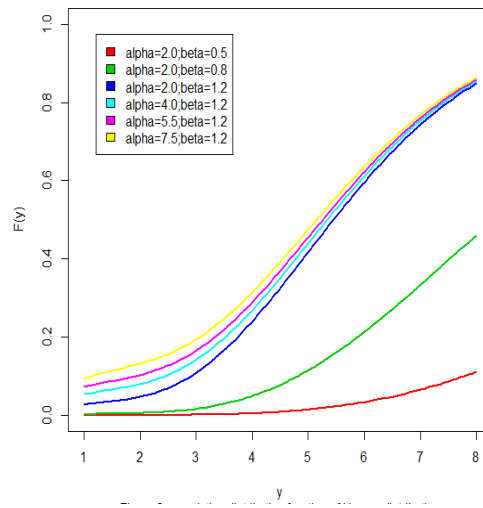
$$F(y, \alpha, \beta) = 1 - \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y}$$

$$; y > 0, \alpha, \beta > 0 \tag{1.2}$$

The graphs of the p.d.f. and c.d.f. of Hamza distribution for different values of parameter are shown in Figs. 1 and 2.



**Fig.1. Probability distribution function of Hamza distribution**



**Fig. 2. Cumulative distribution function of Hamza distribution**

Now, we prove that equation (1.1) follows probability density function

$$\begin{aligned}
 \int_0^{\infty} f(y, \alpha, \beta) &= 1 \\
 \Rightarrow \int_0^{\infty} \frac{\beta^6}{\alpha\beta^5 + 120} \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y} dy \\
 \Rightarrow \frac{\beta^6}{\alpha\beta^5 + 120} \int_0^{\infty} \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y} dy \\
 \Rightarrow \frac{\beta^6}{\alpha\beta^5 + 120} \left\{ \alpha \int_0^{\infty} e^{-\beta y} dy + \frac{\beta}{6} \int_0^{\infty} y^6 e^{-\beta y} dy \right\} \\
 \Rightarrow \frac{\beta^6}{\alpha\beta^5 + 120} \left( \frac{\alpha}{\beta} + \frac{6!}{6\beta^6} \right) = \frac{\beta^6}{\alpha\beta^5 + 120} \left( \frac{\alpha\beta^5 + 120}{\beta^6} \right) = 1
 \end{aligned}$$

Hence, equation (1.1) is following probability density function

## 2 Statistical Properties

### 2.1 Moments of the Hamza distribution

Let  $Y$  be a random variable follows Hamza distribution. Then  $r^{th}$  moment denoted by  $\mu'_r$  is given as

$$\begin{aligned} \mu'_r &= E(Y^r) = \int_0^\infty y^r f(y, \alpha, \beta) dy \\ &= \int_0^\infty y^r \frac{\beta^6}{\alpha\beta^5 + 120} \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y} dy \\ &= \frac{\beta^6}{\alpha\beta^5 + 120} \int_0^\infty \left( \alpha y^r + \frac{\beta}{6} y^{r+6} \right) e^{-\beta y} dy \\ &= \frac{\beta^6}{\alpha\beta^5 + 120} \left[ \frac{\alpha \Gamma(r+1)}{\beta^{r+1}} + \frac{\beta \Gamma(r+7)}{6\beta^{r+7}} \right] \\ \mu'_r &= \frac{1}{\beta^r (\alpha\beta^5 + 120)} \left( \alpha\beta^5 \Gamma(r+1) + \frac{\Gamma(r+7)}{6} \right) \end{aligned}$$

Now substituting  $r = 1, 2, 3, 4$  the first four moments about origin of the Hamza distribution are given as

$$\begin{aligned} \mu'_1 &= \frac{\alpha\beta^5 + 840}{\beta(\alpha\beta^5 + 120)}, \mu'_2 = \frac{2(\alpha\beta^5 + 3360)}{\beta^2(\alpha\beta^5 + 120)} \\ \mu'_3 &= \frac{6(\alpha\beta^5 + 10080)}{\beta^3(\alpha\beta^5 + 120)}, \mu'_4 = \frac{24(\alpha\beta^5 + 25200)}{\beta^4(\alpha\beta^5 + 120)} \end{aligned}$$

Mean:  $\mu = \frac{\alpha\beta^5 + 840}{\beta(\alpha\beta^5 + 120)}$

Variance:  $\sigma^2 = \mu'_2 - (\mu'_1)^2$

$$= \frac{2(\alpha\beta^5 + 3360)}{\beta^2(\alpha\beta^5 + 120)} - \left[ \frac{\alpha\beta^5 + 840}{\beta(\alpha\beta^5 + 120)} \right]^2 = \frac{\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800}{\beta^2(\alpha\beta^5 + 120)^2}$$

The moments about mean of the Hamza distribution are obtained by using relationship between moments about mean and moments about origin

$$\begin{aligned} \mu_2 &= \frac{\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800}{\beta^2(\alpha\beta^5 + 120)^2} \\ \mu_3 &= \frac{2(\alpha^3\beta^{15} + 20520\alpha^2\beta^{10} - 561600\alpha\beta^5 + 527385600)}{\beta^3(\alpha\beta^5 + 120)^3} \\ \mu_4 &= \frac{3 \left( 3\alpha^4\beta^{20} + 29088\alpha^3\beta^{15} + 7689600\alpha^2\beta^{10} \right. \\ &\quad \left. - 4491727200\alpha\beta^5 + 6269926279680 \right)}{\beta^4(\alpha\beta^5 + 120)^4} \end{aligned}$$

The standard deviation (S.D), coefficient of variation (c.v), coefficient of skewness ( $\sqrt{\beta_1}$ ), coefficient of kurtosis ( $\beta_2$ ), index of dispersion ( $\gamma$ ) of Hamza distribution are determined as

$$\sigma = \frac{\sqrt{\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800}}{\beta(\alpha\beta^5 + 120)}$$

$$C.V = \frac{\sigma}{\mu_1} = \frac{\sqrt{\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800}}{\alpha\beta^5 + 840}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{2(\alpha^3\beta^{15} + 20520\alpha^2\beta^{10} - 561600\alpha\beta^5 + 527385600)}{(\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800)^{\frac{3}{2}}}$$

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{3\{3\alpha^4\beta^{20} + 29088\alpha^3\beta^{15} + 7689600\alpha^2\beta^{10}\} - 4491727200\alpha\beta^5 + 6269926279680}{(\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1} = \frac{\alpha^2\beta^{10} + 5280\alpha\beta^5 + 100800}{\beta(\alpha\beta^5 + 120)(\alpha\beta^5 + 840)}$$

### 2.2 Moment generating function of Hamza distribution

Let  $Y$  be a random variable follows Hamza distribution. Then the moment generating function of the distribution denoted by  $M_Y(t)$  is given as

$$\begin{aligned} M_Y(t) &= \int_0^\infty e^{ty} f(y, \alpha, \beta) dy \\ &= \int_0^\infty e^{ty} \frac{\beta^6}{(\alpha\beta^5 + 120)} \left(\alpha + \frac{\beta}{6}y^6\right) e^{-\beta y} dy \\ &= \frac{\beta^6}{(\alpha\beta^5 + 120)} \int_0^\infty \left(\alpha + \frac{\beta}{6}y^6\right) e^{-(\beta-t)y} dy \\ &= \frac{\beta^6}{(\alpha\beta^5 + 120)} \left[ \int_0^\infty \alpha e^{-(\beta-t)y} dy + \frac{\beta}{6} \int_0^\infty y^6 e^{-(\beta-t)y} dy \right] \end{aligned}$$

Using  $\int_0^\infty x^{a-1} e^{-bx} = \frac{\Gamma(a)}{(b)^a}$

$$\begin{aligned} &= \frac{\beta^6}{(\alpha\beta^5 + 120)} \left\{ \alpha \left[ \frac{e^{-(\beta-t)y}}{-(\beta-t)} \right]_0^\infty + \frac{\beta}{6} \frac{\Gamma(7)}{(\beta-t)^7} \right\} \\ &= \frac{\beta^6}{(\alpha\beta^5 + 120)} \left[ \frac{\alpha}{\beta-t} + \frac{\beta}{6} \frac{\Gamma(7)}{(\beta-t)^7} \right] \\ &= \frac{1}{(\alpha\beta^5 + 120)} \left[ \alpha\beta^5 \sum_{k=0}^\infty \left(\frac{t}{\beta}\right)^k + 120 \sum_{k=0}^\infty \binom{k+6}{k} \left(\frac{t}{\beta}\right)^k \right] \\ &= \sum_{k=0}^\infty \frac{6\alpha\beta^5 + (k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}{6(\alpha\beta^5 + 120)} \left(\frac{t}{\beta}\right)^k \end{aligned}$$

### 3 Reliability Measures

Suppose  $Y$  be a continuous random variable with c.d.f  $(y)$ ,  $y \geq 0$ . Then its reliability function which is also called survival function is defined as

$$S(y) = p_r(Y > y) = \int_0^{\infty} f(y) dy = 1 - F(y)$$

Therefore, the survival function for Hamza distribution is given as

$$S(y, \alpha, \beta) = 1 - F(y, \alpha, \beta) = \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y} \quad (3.1)$$

The hazard function of a random variable  $y$  is given as

$$H(y, \alpha, \beta) = \frac{f(y, \alpha, \beta)}{S(y, \alpha, \beta)} \quad (3.2)$$

Using equation (1.1) and equation (3.1) in (3.2), we get

$$H(y, \alpha, \beta) = \frac{6\beta^6 \left( \alpha + \frac{\beta}{6} y^6 \right)}{[6(\alpha\beta^5 + 120) + y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)]}$$

The graph of survival function and Hazard function of Hamza distribution is shown in Figs. 3 and 4.

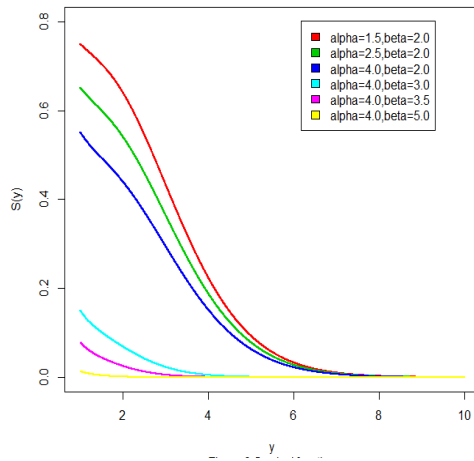


Fig. 3. Survival function

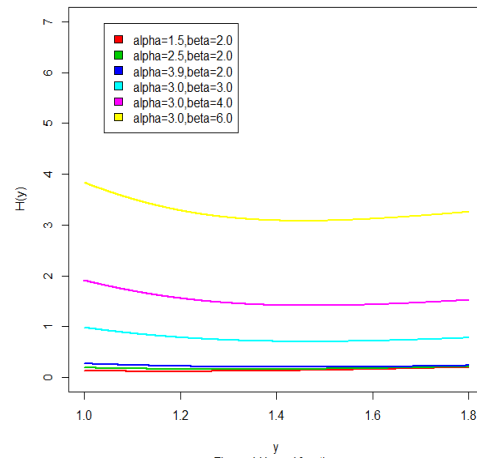


Fig. 4. Hazard function

Also, the reverse hazard function denoted as  $h_r(y, \alpha, \beta)$  can be obtained as

$$h_r(y, \alpha, \beta) = \frac{f(y, \alpha, \beta)}{F(y, \alpha, \beta)} \quad (3.3)$$

using (1.1) and (1.2) in equation (3.3), we get

$$= \frac{6\beta^6 \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y}}{6(\alpha\beta^5 + 120) - [6(\alpha\beta^5 + 120) + y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)]e^{-\beta y}}$$

The mean residual function denoted by  $m(y)$ , and is defined as

$$m(y) = \frac{1}{1 - F(y)} \int_y^\infty 1 - F(t) dt$$

Therefore, the mean residual function of Hamza distribution is given by

$$m(y) = \frac{\{y^6\beta^6 + 12y^5\beta^5 + 90y^4\beta^4 + 480y^3\beta^3 + 1800y^2\beta^2 + 4320y\beta + 6\alpha\beta^5 + 5040\} e^{-\beta y}}{6\beta(\alpha\beta^5 + 120)}$$

We observe that  $H(0) = f(0) = \frac{\alpha\beta^6}{\alpha\beta^5 + 120}$ ,  $m(0) = \mu = \frac{\alpha\beta^5 + 840}{\beta(\alpha\beta^5 + 120)}$

## 4 Stochastic Ordering

Stochastic ordering is a rule by which positive continuous random variables are quantified for their comparative performance. A random variable  $Y$  is said to be smaller than a random variable  $z$  in the

- (1) Stochastic order ( $Y \leq_{st} Z$ ) if  $F_Y(y) \geq F_Z(y)$  for all  $y$
- (2) Hazard rate order ( $Y \leq_{hr} Z$ ) if  $H_Y(y) \geq H_Z(y)$  for all  $y$
- (3) Mean residual life order ( $Y \leq_{mrl} Z$ ) if  $m_Y(y) \geq m_Z(y)$  for all  $y$
- (4) Likelihood ratio order ( $Y \leq_{lr} Z$ ) if  $\frac{f_Y(y)}{f_Z(y)}$  decreases in  $y$

The below result by Shaked and Shanthikumar [10] are famous for demonstrating stochastic ordering of distributions

$$Y \leq_{lr} Z \Rightarrow Y \leq_{hr} Z \Rightarrow Y \leq_{mrl} Z \\ \Downarrow Y \leq_{st} Z$$

The following theorem shows that Hamza distribution is ordered with respect to the strongest ‘likelihood ratio ordering’:

**Theorem:** Suppose  $Y \sim H(\alpha_1, \beta_1)$  and  $Z \sim H(\alpha_2, \beta_2)$ . If  $\alpha_1 = \alpha_2$  and  $\beta_1 > \beta_2$  or  $\beta_1 = \beta_2$  and  $\alpha_1 > \alpha_2$ . Then  $Y \leq_{lr} Z$ ,  $Y \leq_{hr} Z$ ,  $Y \leq_{mrl} Z$ , and  $Y \leq_{st} Z$ .

**Proof:** We have

$$\frac{f_Y(y, \alpha_1, \beta_1)}{f_Z(y, \alpha_2, \beta_2)} = \frac{\beta_1^6(\alpha_2\beta_2^5 + 120)}{\beta_2^6(\alpha_1\beta_1^5 + 120)} \left[ \frac{6\alpha_1 + \beta_1 y^6}{6\alpha_2 + \beta_2 y^6} \right] e^{-(\beta_1 - \beta_2)y} ; y > 0$$

Applying log, we have

$$\log \frac{f_Y(y, \alpha_1, \beta_1)}{f_Z(y, \alpha_2, \beta_2)} = \log \left[ \frac{\beta_1^6(\alpha_2\beta_2^5 + 120)}{\beta_2^6(\alpha_1\beta_1^5 + 120)} \right] + \log \left[ \frac{6\alpha_1 + \beta_1 y^6}{6\alpha_2 + \beta_2 y^6} \right] - (\beta_1 - \beta_2)y$$

Differentiating w.r.t  $y$ , we get

$$\frac{d}{dy} \log \frac{f_Y(y, \alpha_1, \beta_1)}{f_Z(y, \alpha_2, \beta_2)} = \frac{36(\alpha_2\beta_1 - \alpha_1\beta_2)}{(6\alpha_1 + \beta_1 y^6)(6\alpha_2 + \beta_2 y^6)} - (\beta_1 - \beta_2)$$

Thus if  $\alpha_1 = \alpha_2$  and  $\beta_1 > \beta_2$  or  $\beta_1 = \beta_2$  and  $\alpha_1 > \alpha_2$ , we have

$$\frac{d}{dy} \log \frac{f_Y(y, \alpha_1, \beta_1)}{f_Z(y, \alpha_2, \beta_2)} < 0$$

Which implies that  $Y \leq_{lr} Z$  and hence  $Y \leq_{hr} Z, Y \leq_{mrl} Z,$  and  $Y \leq_{st} Z$

### 5 Renyi Entropy

If  $Y$  is a continuous random variable having probability density function  $f(y, \alpha, \beta)$ , then Renyi entropy is defined as

$$T_R(\delta) = \frac{1}{1-\delta} \log \left\{ \int_0^\infty f^\delta(y) dy \right\}$$

where  $\delta > 0$  and  $\delta \neq 1$

Thus, the Renyi entropy for Hamza distribution (1.1), is given as

$$\begin{aligned} T_R(\delta) &= \frac{1}{1-\delta} \log \left\{ \int_0^\infty \left[ \frac{\beta^6}{\alpha\beta^5 + 120} \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y} \right]^\delta dy \right\} \\ &= \frac{1}{1-\delta} \log \left\{ \frac{\beta^{\delta 6}}{(\alpha\beta^5 + 120)^\delta} \int_0^\infty \left( \alpha + \frac{\beta}{6} y^6 \right)^\delta e^{-\beta \delta y} dy \right\} \\ &= \frac{1}{1-\delta} \log \left\{ \frac{\alpha^\delta \beta^{\delta 6}}{(\alpha\beta^5 + 120)^\delta} \int_0^\infty \left( 1 + \frac{\beta}{6\alpha} y^6 \right)^\delta e^{-\beta \delta y} dy \right\} \\ &= \frac{1}{1-\delta} \log \left\{ \sum_{r=0}^\infty \binom{\delta}{r} \frac{\alpha^{\delta-r} \beta^{6\delta+r}}{6^r (\alpha\beta^5 + 120)^\delta} \int_0^\infty y^{6r} e^{-\beta \delta y} dy \right\} \\ &= \frac{1}{1-\delta} \log \left\{ \sum_{i=0}^\infty \binom{\delta}{r} \frac{\alpha^{\delta-r} \beta^{6\delta+r} \Gamma(6r+1)}{6^r (\alpha\beta^5 + 120)^\delta (\beta \delta)^{6r+1}} \right\} \\ &= \frac{1}{1-\delta} \log \left\{ \sum_{i=0}^\infty \binom{\delta}{r} \frac{\alpha^{\delta-r} \beta^{6\delta-(5r+1)} r \Gamma(6r)}{6^{r-1} (\alpha\beta^5 + 120)^\delta \delta^{6r+1}} \right\} \end{aligned}$$

### 6 Mean Deviation from Mean of Hamza Distribution

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations. Let  $Y$  be a random variable from Hamza distribution with mean  $\mu$  then the mean deviation from mean is defined as.

$$\begin{aligned} D(\mu) &= E(|Y - \mu|) = \int_0^\infty |Y - \mu| f(y) dy \\ &= \int_0^\mu (\mu - y) f(y) dy + \int_\mu^\infty (y - \mu) f(y) dy \\ &= \mu \int_0^\mu f(y) dy - \int_0^\mu y f(y) dy + \int_\mu^\infty y f(y) dy - \int_\mu^\infty \mu f(y) dy \\ &= \mu F(\mu) - \int_0^\mu y f(y) dy - \mu [1 - F(\mu)] + \int_\mu^\infty y f(y) dx \\ &= 2\mu F(\mu) - 2 \int_0^\mu y f(y) dy \end{aligned} \tag{6.1}$$



Now

$$\int_0^\mu yf(y)dy = \frac{\beta^6}{\alpha\beta^5 + 120} \int_0^\mu \left(\alpha + \frac{\beta}{6}y^6\right) e^{-\beta y} dy$$

After solving the integral, we get

$$= \mu - \frac{\left\{ \mu^7\beta^7 + 7\mu^6\beta^6 + 42\mu^5\beta^5 + 210\mu^4\beta^4 + 840\mu^3\beta^3 + 2520\mu^2\beta^2 + \mu\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta\mu}}{6\beta(\alpha\beta^5 + 120)} \quad (6.2)$$

Now substituting equation (6.2) in equation (6.1), we get

$$D(\mu) = \frac{\left\{ \mu^6\beta^6 + 12\mu^5\beta^5 + 90\mu^4\beta^4 + 480\mu^3\beta^3 + 1800\mu^2\beta^2 + 4320\mu\beta + \mu\alpha\beta(1 - 6\beta^5) + \alpha + 5040 \right\} e^{-\beta\mu}}{3\beta(\alpha\beta^5 + 120)}$$

## 7 Mean Deviation from Median of Hamza Distribution

Let  $Y$  be a random variable from Hamza distribution with median  $M$  then the mean deviation from median is defined as.

$$\begin{aligned} D(M) &= E(|Y - M|) = \int_0^M (M - y) dy + \int_M^\infty (y - M) dy \\ &= MF(M) - \int_0^M yf(x) dy - M[1 - F(M)] + \int_M^\infty yf(y) dy \\ &= \mu - 2 \int_0^M yf(y) dy \end{aligned} \quad (7.1)$$

Now

$$\int_0^M yf(y) dy = \frac{\beta^6}{\alpha\beta^5 + 120} \int_0^M \left(\alpha + \frac{\beta}{6}y^6\right) e^{-\beta y} dy$$

After solving the integral, we get

$$= \mu - \frac{\left\{ M^7\beta^7 + 7M^6\beta^6 + 42M^5\beta^5 + 210M^4\beta^4 + 840M^3\beta^3 + 2520M^2\beta^2 + M\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta M}}{6\beta(\alpha\beta^5 + 120)} \quad (7.2)$$

Now substituting equation (7.2) in equation (7.1), we get

$$D(M) = \frac{\left\{ M^7\beta^7 + 7M^6\beta^6 + 42M^5\beta^5 + 210M^4\beta^4 + 840M^3\beta^3 + 2520M^2\beta^2 + M\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta M}}{6\beta(\alpha\beta^5 + 120)} - \mu$$

## 8 Bonferroni and Lorenz Curves

In economics the relation between poverty and economy is well studied by using Bonferroni and Lorenz curves [11]. Besides that these curves have been used in different fields such as reliability, insurance and biomedicine.

The Bonferroni curve,  $B(s)$  is given as.

$$B(s) = \frac{1}{s\mu} \int_0^t y f(y) dy \tag{8.1}$$

Or

$$B(s) = \frac{1}{s\mu} \int_0^s F^{-1}(y) dy$$

And Lorenz curve,  $L(s)$  is given as.

$$L(s) = \frac{1}{\mu} \int_0^t y f(y) dy \tag{8.2}$$

Or

$$L(s) = \frac{1}{\mu} \int_0^s F^{-1}(y) dy$$

Where  $E(X) = \mu$  and  $t = F^{-1}(s)$ .

Now

$$\int_0^t y f(y) dy = \mu - \frac{\left\{ t^7 \beta^7 + 7t^6 \beta^6 + 42t^5 \beta^5 + 210t^4 \beta^4 + 840t^3 \beta^3 + 2520t^2 \beta^2 + t\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta t}}{6\beta(\alpha\beta^5 + 120)} \tag{8.3}$$

Substituting equation (8.3) in equations (8.1) and (8.2), we get

$$B(s) = \frac{1}{s} \left[ 1 - \frac{\left\{ t^7 \beta^7 + 7t^6 \beta^6 + 42t^5 \beta^5 + 210t^4 \beta^4 + 840t^3 \beta^3 + 2520t^2 \beta^2 + t\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta t}}{6\beta(\alpha\beta^5 + 120)\mu} \right]$$

And

$$L(s) = \left[ 1 - \frac{\left\{ t^7 \beta^7 + 7t^6 \beta^6 + 42t^5 \beta^5 + 210t^4 \beta^4 + 840t^3 \beta^3 + 2520t^2 \beta^2 + t\beta(\alpha + 5040) + \alpha + 5040 \right\} e^{-\beta t}}{6\beta(\alpha\beta^5 + 120)\mu} \right]$$

## 9 Order Statistics of Hamza Distribution

Let us consider  $Y_1, Y_2 \dots Y_n$  be random sample of sample size  $n$  from Hamza distribution with pdf (1.1) and cdf (1.2). Then the pdf of  $k^{th}$  order statistics is given by

$$f_{Y^{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} [F(y)]^{k-1} [1 - F(y)]^{n-k} f(y) \quad , k = 1, 2, 3, \dots, n \tag{9.1}$$

Now substituting the equation (1.1) and (1.2) in equation (8.1), we obtain the  $k^{th}$  order statistics as

$$f_{Y_{(k)}}(y) = \frac{n! \beta^6 \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y}}{(k-1)! (n-k)! (\alpha \beta^5 + 120)}$$

$$\left\{ 1 - \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y} \right\}^{k-1}$$

$$\left\{ \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y} \right\}^{n-k} \quad (9.2)$$

The pdf of first order statistics  $Y_1$  is given by

$$f_{Y_{(1)}}(y) = \frac{n\beta^6 \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y}}{(\alpha\beta^5 + 120)}$$

$$\left\{ \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y} \right\}^{n-1}$$

And pdf of the  $n^{th}$  order statistics  $Y_n$  is given by

$$f_{Y_{(n)}}(y) = \frac{n\beta^6 \left( \alpha + \frac{\beta}{6} y^6 \right) e^{-\beta y}}{(\alpha\beta^5 + 120)}$$

$$\left\{ 1 - \left[ 1 + \frac{y\beta(y^5\beta^5 + 6y^4\beta^4 + 30y^3\beta^3 + 120y^2\beta^2 + 360y\beta + 720)}{6(\alpha\beta^5 + 120)} \right] e^{-\beta y} \right\}^{n-1}$$

## 10 Estimation of Parameters of Hamza Distribution

Suppose  $Y_1, Y_2, Y_3, \dots, Y_n$  be random samples of size  $n$  from Hamza distribution. Then the likelihood function of Hamza distribution is given as.

$$l = \prod_{i=1}^n f(y_i, \alpha, \beta)$$

$$= \prod_{i=1}^n \left\{ \frac{\beta^6}{\alpha\beta^5 + 120} \left( \alpha + \frac{\beta}{6} y_i^6 \right) e^{-\beta y_i} \right\}$$

$$= \left( \frac{\beta^6}{\alpha\beta^5 + 120} \right)^n \prod_{i=1}^n \left( \alpha + \frac{\beta}{6} y_i^6 \right) e^{-\beta \sum_{i=1}^n y_i}$$

The log likelihood function is given by

$$\log l = 6n \log \beta - n \log(\alpha\beta^5 + 120) + \sum_{i=1}^n \log \left( \alpha + \frac{\beta}{6} y_i^6 \right) - \beta \sum_{i=1}^n y_i \quad (10.1)$$

Now, differentiate(10.1) partially w.r.t the parameters  $\alpha$  and  $\beta$ , we get

$$\frac{\partial \log l}{\partial \alpha} = \frac{-n\beta^5}{\alpha\beta^5 + 120} + \sum_{i=1}^n \frac{6}{(6\alpha + \beta y_i^6)}$$

$$\frac{\partial \log l}{\partial \beta} = \frac{6n}{\beta} - \frac{5n\alpha\beta^4}{\alpha\beta^5 + 120} + \sum_{i=1}^n \frac{y_i^6}{(6\alpha + \beta y_i^6)} - n\bar{y}$$

Now, setting  $\frac{\partial \log l}{\partial \alpha} = 0$  and  $\frac{\partial \log l}{\partial \beta} = 0$ , we get

$$\frac{-n\beta^5}{\alpha\beta^5 + 120} + \sum_{i=1}^n \frac{6}{(6\alpha + \beta y_i^6)} = 0 \tag{10.2}$$

$$\frac{6n}{\beta} - \frac{5n\alpha\beta^4}{\alpha\beta^5 + 120} + \sum_{i=1}^n \frac{y_i^6}{(6\alpha + \beta y_i^6)} - n\bar{y} = 0 \tag{10.3}$$

It is clear that equations (10.2) and (10.3), are non-linear equations cannot be solved explicitly for  $\alpha$  and  $\beta$ . In order to find the value of  $\alpha$  and  $\beta$  it is imperative to apply iterative methods. The MLE of the parameters denoted as  $\hat{\theta}(\hat{\alpha}, \hat{\beta})$  of  $\theta(\alpha, \beta)$  can be obtained by using Newton-Raphson method, bisection method, secant method etc.

Since the MLE of  $\hat{\theta}$  follows asymptotically normal distribution which is given as

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow N(0, I^{-1}(\theta)) \tag{10.4}$$

Where  $I^{-1}(\theta)$  is the limiting variance-covariance matrix of  $\hat{\theta}$  and  $I(\theta)$  is a  $2 \times 2$  Fisher information matrix I,e

$$I(\theta) = -\frac{1}{n} \begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \beta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial^2 \beta}\right) \end{bmatrix}$$

Where

$$\begin{aligned} \frac{\partial^2 \log l}{\partial^2 \alpha} &= -\left\{ \frac{n\beta^{10}}{(\alpha\beta^5 + 120)} + \sum_{i=1}^n \frac{36}{(6\alpha + \beta y_i^6)} \right\} \\ \frac{\partial^2 \log l}{\partial^2 \beta} &= \frac{-6n}{\beta^2} + \frac{n(5\alpha^2\beta^8 - 2400\alpha\beta^3)}{(\alpha\beta^5 + 120)^2} - \sum_{i=1}^n \frac{y_i^{12}}{(6\alpha + \beta y_i^6)^2} \\ \frac{\partial^2 \log l}{\partial \alpha \partial \beta} &= \frac{\partial^2 \log l}{\partial \beta \partial \alpha} = \frac{-600n\beta^4}{(\alpha\beta^5 + 120)} - \sum_{i=1}^n \frac{6y_i^6}{(6\alpha + \beta y_i^6)^2} \end{aligned}$$

Hence the approximate  $100(1 - \psi)\%$  confidence interval for  $\alpha$  and  $\beta$  are respectively given by

$$\hat{\alpha} \pm z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\theta})}, \hat{\beta} \pm z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\theta})}$$

Where  $z_{\frac{\psi}{2}}$  is the  $\psi^{th}$  percentile of the standard distribution

## 11 Data Analysis

**Data set 1:** In this section, we illustrate the performance of the newly established model by using two real data sets. And we show that established distribution perform better than Ishita distribution, Pranav distribution, Lindley distribution and Exponential distribution. The data set is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by Cordeiro and Lemonte [12] and Al-Aqtash et al. [13]. The data is as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

In order to compare the above distribution models, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion). Among the above distributions, the better distribution is considered having lesser values of AIC, AICC, and BIC.

**Table 11.1. MLEs and criteria for comparison**

Model	Parameter estimates	S.E	-2logL	AIC	AICC	BIC
Hamza Distribution	$\hat{\alpha} = 0.0214$ $\hat{\beta} = 2.4994$	0.02385 0.11855	178.9673	182.967	183.2223	186.7913
Ishita Disribution	$\hat{\beta} = 0.6391$	0.0488	191.2379	193.2379	193.3629	195.1499
Pranav Distribution	$\hat{\beta} = 1.1548$	0.05832	221.0816	223.0816	223.2066	224.9936
Lindley Distribution	$\hat{\beta} = 0.5902$	0.05324	244.7681	246.7681	246.8931	248.6801
Exponential Distribution	$\hat{\beta} = 0.3623$	0.0446	265.987	267.987	268.112	269.899

**Data set 2:** The data set consists of observations on breaking stress of carbon fibers (in Gba). The data has been previously used by Cordeiro and Lemonte [12] and Al-Aqtash et al. [13] the data is given as:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

**Table 11.2. MLEs and criteria for comparison**

Model	Parameter estimates	S.E	-2log L	AIC	AICC	BIC
Hamza Distribution	$\hat{\alpha} = 0.0105$ $\hat{\beta} = 2.6444$	0.0124 0.1017	285.6293	289.6293	289.753	294.8396

Model	Parameter estimates	S.E	-2log L	AIC	AICC	BIC
Ishita Distribution	$\hat{\beta} = 0.6658$	0.0408	295.846	297.846	297.9072	300.4511
Pranav Distribution	$\hat{\beta} = 1.1915$	0.04844	339.0479	341.0479	341.1091	343.653
Lindley Distribution	$\hat{\beta} = 0.6173$	0.0453	363.5069	365.5069	365.5681	368.112
Exponential Distribution	$\hat{\beta} = 0.3814$	0.3814	392.7417	394.7417	394.8029	399.3468

From Tables 11.1 and 11.2, it has been observed that the Hamza distribution have the lesser AIC, AICC,  $-2 \log L$  and BIC values as compared to Ishita, Pranav, Lindley and exponential distribution. Now, we can conclude that Hamza distribution provides better fit than compared ones.

## 12 Concluding Remarks

In the present paper a new life time distribution has been established named “Hamza distribution” its several properties including moments, coefficient of variation, skewness, kurtosis, index of dispersion, moment generating function, survival function, hazard rate function, reverse hazard function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy have been discussed. The parameters of the distribution has been estimated by known method of maximum likelihood estimator. Finally the performance of the model has been examined through two data sets and compared with Ishita, Pranav, Lindley and exponential distribution. Result shows that Humza distribution gives an adequate fit for the data sets.

## Competing Interests

Authors have declared that no competing interests exist.

## References

- [1] Lindley DV. Fiducial distributions and Bayes’ theorem, Journal of Royal Statistical Society, Series B. 1958; 20:102-107.
- [2] Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its applications. Mathematics Computing and Simulation. 2008;78:493-506.
- [3] Zakerzadeh H, Dolati A. Generalized Lindley distribution. Journal of Mathematical Extension. 2009; 3(2):13-25.
- [4] Nadarajah S, Bakouch HS, Tahmasbi RA. Generalized Lindley distribution. Sankhya Series B. 2011;73:331-359.
- [5] Merovci F. Transmuted Lindley distribution. International Journal of Open Sprblems in Computer Science and Mathematics. 2013;6:63-72.
- [6] Sharma V, Singh S, Singh U, Agiwal V. The inverse Lindley distribution – a stress-strength reliability model with applications to head and neck cancer data, Journal of Industrial & Production Engineering. 2015;32(3):1623-173.

- [7] Shaker R, Shukla KK. Ishita distribution and its Applications. BBIJ. 2017;5(2):1-9.
- [8] Shukla KK. Pranav distribution with properties and its applications. BBIJ. 2018;7(3):244-253.
- [9] Aijaz A, Ahmad A, Tripathi R. Transmuted inverse Lindley distribution: Statistical properties Applications, Science, Technology and Development. 2020;9(7):1-10.
- [10] Shaked M, Shanthikumar JG. Stochastic orders and their applications. Academic Press, New York; 1994.
- [11] Bonferroni CE. Elementi di Statiscagnerarale, Seeber, Firenze; 1930.
- [12] Cordeiro GM, Lemonte AJ. The  $\beta$ -Birnbaum-Saunders distribution: An improved distribution for fatigue life modeling”, Computational Statistics and Data Analysis. 2011;55:1445-1461.
- [13] Al-Aqtash R, Lee C, Famoye F. Gumbel-weibull distribution: Properties and applications. Journal of Modern Applied Statistical Methods. 2014;13:201-225.

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