Demonstration of the Importance of Factoring in Solving a Fundamental Equation Involving the Riemann Zeta Function

Bibek Baran Nag[1*]

[1]*Independent Researcher, UK.*

Original Research Article

Abstract

The author presents a simple approach which can be used to tackle various special cases of some well-known problems involving zeta functions. A self-contained argument, which requires only basic prerequisite mathematical knowledge, is used to furnish a new proof of a result involving the Riemann zeta function which can help in settling more general conjectures.

1 Introduction

The most important mathematical conjectures have acquired such inapproachable status that attempts at their resolutions are discouraged. Having constantly refused to accept the existence

---

*Corresponding author: E-mail: bibek010101@gmail.com;

of its proof, the present author has continually held the opinion that it is irresponsible to promulgate the idea that the Riemann Hypothesis should be regarded as a genuine prized problem of mathematics. This point of view provided motivation for this piece of work. An account of the Riemann zeta function and its connections to the distribution of primes is provided in [1]. Hardy [2] proved that the Riemann Hypothesis is true for infinitely many zeros. More recently, Levinson [3] and Conrey [4] have proved that the Riemann Hypothesis is true for significant proportions of the zeros.

Define s to be a general complex number given by $s = \sigma + it$, where σ and t are both real and $0 < \sigma < 1$. It is known [5] that the Riemann zeta function in the region bounded by $0 < \sigma < 1$ may be expressed in terms of the convergent alternating zeta function, so that

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \tag{1.1}$$

which is easily established (see [5]) by multiplying both its sides by $(1 - 2^{1-s})$ in a manner similar to a standard method of determining the Euler product for the Riemann zeta function (which is stated and proved in a different way in [6]). A variant of this method is used to derive Lemma 2. The preceding equation is used to determine fundamental results which are presented in the next section.

2 Analysis

The following result is established by using exponential notation.

Lemma 1. *The complex conjugate of $\zeta(\sigma + it)$ is given by $\zeta(\sigma - it)$.*

Proof. By substituting $s = \sigma + it$ into (1.1), it is clear that

$$\zeta(\sigma + it) = \frac{1}{(1 - 2^{1-\sigma_1} \cdot 2^{-it_1})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\sigma_1} \cdot n^{it_1}}. \tag{2.1}$$

Note that

$$n^{\pm it} = e^{\pm it \ln n}, \tag{2.2}$$

and also, in particular, that

$$2^{\pm it} = e^{\pm it \ln 2}. \tag{2.3}$$

By noting that the exponential functions can be replaced by trigonometric functions, the statement of the theorem follows from substituting both (2.2) and (2.3) into (2.1). □

Lemma 2. *The convergent alternating zeta function in (1.1) can be expressed as an infinite product given by*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1 - 2^{s-1}}{1 - 2^s} \prod_{j=2}^{\infty} \frac{2}{1 - p_j^{-s}}, \tag{2.4}$$

where p_j is a unique prime which is larger than at most $j - 1$ other primes.

Proof. The convergent alternating zeta function can be rewritten by separating its terms with denominators which are powers of odd integers from those with denominators which are powers of even integers, so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}. \tag{2.5}$$

Multiplying the series $\sum_{n=0}^{\infty}(2n+1)^{-s}$ by $(1-p_j^{-s})$ removes all its terms with denominators which are powers of multiples of p_j, where $j \geq 2$. Multiplying the series $\sum_{n=1}^{\infty}(2n)^{-s}$ by $(1-p_j^{-s})$ removes every single one of its terms with a denominator which is a multiple of p_j. After repeating this procedure for all distinct successive p_j, it follows that multiplying the series $\sum_{n=0}^{\infty}(2n+1)^{-s}$ by $\prod_{j=2}(1-p_j^{-s})$ removes all its terms except its first which is equal to 1. Similarly, multiplying the series $\sum_{n=1}^{\infty}(2n)^{-s}$ by $\prod_{j=2}(1-p_j^{-s})$ removes all its terms except those with denominators which are perfect powers of 2^s. It follows from the last two sentences that (2.5) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \prod_{j=2}^{\infty} \frac{1}{1-p_j^{-s}} \left(1 - \sum_{n=1}^{\infty} \frac{1}{2^{ns}}\right). \tag{2.6}$$

Then, multiplying the series $\sum_{n=1}^{\infty} 2^{-ns}$ by $(1-2^{-s})$ removes all its terms except the first one which is given by 2^{-s}, so that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \prod_{j=2}^{\infty} \frac{1}{1-p_j^{-s}} \left(1 - \frac{2^{-s}}{(1-2^{-s})}\right), \tag{2.7}$$

which implies that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \prod_{j=2}^{\infty} \frac{1}{1-p_j^{-s}} \left(1 - \frac{1}{2^s(1-2^{-s})}\right), \tag{2.8}$$

i.e.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \prod_{j=2}^{\infty} \frac{1}{1-p_j^{-s}} \left(1 + \frac{1}{1-2^s}\right), \tag{2.9}$$

from which the desired result follows easily. $\square$

Theorem 1. *If $\zeta(s) = 0$, then $\sigma = \frac{1}{2}$.*

Proof. It is known [7] that there exists some positive t_0 such that $\zeta\left(\frac{1}{2} \pm it_0\right) = 0$. By using Lemma 1 and (1.1), it follows that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{1}{2}\pm it_0}} = 0$ for any t_0. It is easily seen that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ is nonzero if $t = 0$ which, together with Lemma 1, implies that any t_0 is positive. Then, by considering Lemma 1, any $\zeta\left(\frac{1}{2} \pm it_0\right)$ can be factored out of $\zeta(s)$ and any $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{1}{2}\pm it_0}}$ can be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$. It follows from an application of Lemma 2 that $(1-2^s)^{-1}$ can be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ which itself, by considering (1.1), can be factored out of $\zeta(s)$. It is easily seen from the last statement that any $\left(1 - 2^{\left(\frac{1}{2}\pm it_0\right)}\right)^{-1}$ can be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{1}{2}\pm it_0}}$. It follows from the last three sentences that any $\left(1 - 2^{\left(\frac{1}{2}\pm it_0\right)}\right)^{-1}$ can be factored out of both $\zeta(s)$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$. By using the fact that the difference of two squares can be factored, it is immediately apparent that $(1-2^s)$ can be expressed as $\left(1 - 2^{\frac{\sigma+it}{2}}\right)\left(1 + 2^{\frac{\sigma+it}{2}}\right)$, where $0 < \sigma < 1$. Since $\sigma \neq 1$, it follows that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$, where t_1 is any nonzero real number, cannot be equal to $\left(1 - 2^{\frac{\sigma+it}{2}}\right)$. Since

$$-1 = e^{\pm\left(\frac{i\pi}{\ln 2}\right)\cdot \ln 2} = e^{\ln 2^{\pm \frac{i\pi}{\ln 2}}} = 2^{\pm \frac{i\pi}{\ln 2}}, \tag{2.10}$$

it follows that

$$\left(1 + 2^{\frac{\sigma+it}{2}}\right) = \left(1 - 2^{\frac{\sigma+it}{2}} \cdot 2^{\pm \frac{i\pi}{\ln 2}}\right), \tag{2.11}$$

and

$$\left(1 + 2^{(\sigma+it)\cdot 2^{-r}}\right) = \left(1 - 2^{(\sigma+it)\cdot 2^{-r}} \cdot 2^{\pm \frac{i\pi}{\ln 2}}\right), \tag{2.12}$$

where r is any positive integer. Since $\sigma \neq 1$, it follows from (2.11) that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$ cannot be equal to $\left(1 + 2^{\frac{\sigma+it}{2}}\right)$. Also $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$ cannot be factored out of $\left(1 + 2^{\frac{\sigma-it}{2}}\right)$. By repeating the same procedure of factoring the difference of two squares, it can be seen that $\left(1 - 2^{\frac{\sigma+it}{2}}\right)$ can be expressed as $\left(1 - 2^{\frac{\sigma+it}{4}}\right)\left(1 + 2^{\frac{\sigma+it}{4}}\right)$. Since $\sigma < 1$, it follows that $\left(1 - 2^{\left(\frac{1}{2}-it_1\right)}\right)$ also cannot be equal to $\left(1 - 2^{\frac{\sigma+it}{4}}\right)$ or indeed to $\left(1 - 2^{(\sigma+it)\cdot 2^{-r}}\right)$ for any positive integer r. Since $\sigma < 1$, it follows from (2.12) that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$ also cannot be equal to $\left(1 + 2^{(\sigma+it)\cdot 2^{-r}}\right)$. Also $\left(1 \pm 2^{\left(\frac{1}{2}+it_1\right)\cdot 2^{-m}}\right)$, where m is any positive integer, cannot be factored out of $\left(1 + 2^{(\sigma+it)\cdot 2^{-r}}\right)$, unless $\left(1 + 2^{\left(\frac{1}{2}+it_1\right)\cdot 2^{-m}}\right)$ is equal to the latter expression in the instance that $\sigma = \frac{1}{2}$, $t = t_1$ and $m = r$. It follows from the last statement that unless $\sigma = \frac{1}{2}$ and $t = t_1$, neither $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$ nor any quantity that can be factored out of it can be factored out of $\left(1 + 2^{(\sigma+it)\cdot 2^{-r}}\right)$. Hence, having established that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)$ cannot be equal to $\left(1 - 2^{(\sigma+it)\cdot 2^{-r}}\right)$ for any positive integer r, it is evident (from repeatedly applying the concept of the difference of two squares in the manner described before) that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)^{-1}$ cannot be factored out of $\left(1 \pm 2^{\frac{\sigma+it}{2}}\right)^{-1}$. It follows that $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)^{-1}$ can be factored out of $\left((1 - 2^{\sigma+it})^{-1}\right.$ only if $\sigma = \frac{1}{2}$ and $t = t_1$. Suppose that α is any positive integer and that there exists some quantity $F_\alpha^\pm$ such that

$$\left(1 \pm 2^{\left(\frac{1}{2^\alpha}+i\frac{t_1}{2^{\alpha-1}}\right)}\right)^{-1} \cdot F_\alpha^\pm = (1 - 2^{s-1}). \tag{2.13}$$

Then

$$F_\alpha^\pm = \left(1 \pm 2^{\left(\frac{1}{2^\alpha}+i\frac{t_1}{2^{\alpha-1}}\right)}\right) \cdot (1 - 2^{s-1}), \tag{2.14}$$

from which, because $(1 - 2^{s-1})$ is not equal to 1 and can be factored out of $F_\alpha^\pm$, it is apparent that $F_\alpha^\pm$ cannot be factored out of $(1 - 2^{s-1})$. By considering this last statement with (2.13), it follows that any factor of $\left(1 - 2^{\left(\frac{1}{2}+it_1\right)}\right)^{-1}$ (including itself), being of the form $\left(1 \pm 2^{\left(\frac{1}{2^\alpha}+i\frac{t_1}{2^{\alpha-1}}\right)}\right)^{-1}$ by considering the difference of pairs of squares, cannot be factored out of $(1 - 2^{s-1})$. Having established earlier that any $\left(1 - 2^{\left(\frac{1}{2}\pm it_0\right)}\right)^{-1}$ can be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$, it follows with an application of Lemma 2 that it can also be entirely factored out of only $(1 - 2^s)^{-1}$ in (2.4) but not out of its factors given by $\left(1 \pm 2^{\frac{s}{2}}\right)^{-1}$, in which case $\sigma = \frac{1}{2}$ and $t = \pm t_0$. It has been proved that any $\left(1 - 2^{\left(\frac{1}{2}\pm it_0\right)}\right)^{-1}$ can be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ in (1.1) only if $\sigma = \frac{1}{2}$ and $t = \pm t_0$. It is clear from (2.4) that $(s - k)$, where k is any complex number with a real part which is both positive and strictly less than 1, cannot be factored out of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$. Then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 0$ only if $\sigma = \frac{1}{2}$ and $t = \pm t_0$. By applying (1.1), the desired result follows from the last statement. $\square$

3   Conclusion

The importance of factoring in solving a fundamental equation involving the Riemann zeta function has been demonstrated. These elementary arguments can be easily extended to tackle other conjectures such as the Generalized Riemann Hypothesis.

Acknowledgement

We thank the Editor and the referee for their comments.

Competing Interests

The author has declared that no competing interests exist.

References

[1] Ingham AE. The distribution of prime numbers. Cambridge University Press. 1932;30:25-67.

[2] Hardy GH. Sur les zéros de la fonction $\zeta(s)$ de Riemann. Comptes Rendus de l'Académie des Sciences. 1914;158:1012-1014.

[3] Levinson N. More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$. Advances in Mathematics. 1974;13:383-436.

[4] Conrey JB. More than two fifths of the zeros of the Riemann zeta function are on the critical line. Journal für die Reine und Angewandte Mathematik. 1989;399:1-26.

[5] Stopple J. A primer of analytic number theory. 1st Edition, Cambridge University Press. 2003;195-196.

[6] Hardy GH, Wright EM. An introduction to the theory of numbers. 4th Edition, Oxford University Press. 1960;246-247.

[7] Edwards HM. Riemann's zeta function. 1st Edition, Academic Press Inc. 1974;121-122.