Open Journal of
Mathematical Analysis

PSR Press

Article

Fixed point theorems for generalized (ψ, φ, F)-contraction type mappings in b-metric spaces with applications

Taieb Hamaizia

Laboratory of Dynamical Systems and Control, Department of Mathematics and Informatics, Oum El Bouaghi University, 04000, Algeria.; tayeb042000@yahoo.fr

Communicated by: Absar ul Haq
Received: 31 December 2020; Accepted: 22 February 2021; Published: 28 February 2021.

Abstract: The purpose of this paper is to prove a fixed point theorem for C-class functions in complete b-metric spaces. Moreover, the solution of the integral equation is obtained using our main result.

Keywords: b-metric space, fixed point, altering distance function.

MSC: 47H10, 54H25.

---

1. Introduction

The concept of b-metric spaces was first introduced by Bakhtin [1] and Czerwik [2] and utilized for $s = 2$ and for an arbitrary $s \geq 1$ to prove some generalizations of Banach's fixed point theorem [3]. In 2010, Khamsi and Hussain [4] reintroduced the notion of b-metric and called it a metric-type. Afterwards, several authors proved fixed and common fixed point theorems for single-valued mappings in b-metric spaces, see [5–16].

In this paper, we introduce the definition of C-class functions and (ψ, φ, F)-contraction type mappings where ψ is the altering distance function and φ is the ultra altering distance function. The unique fixed point theorem for self mapping in the setting of b-complete metric spaces is proven. In the end of paper, we apply our main result to approximating the solution of the Fredholm integral equation.

In the sequel, we always denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ the set of positive integers, real numbers, and nonnegative real numbers, respectively. The following definitions, notations, basic lemma and remarks will be needed in the sequel.

Definition 1. [1] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied;

(bm-1) $d(x,y) = 0 \iff x = y$,
(bm-2) $d(x,y) = d(y,x)$,
(bm-3) $d(x,y) \leq s(d(x,z) + d(z,y))$.

The pair (X, d) is called a b-metric space with a coefficient s.

Every metric space is a b-metric space with $s = 1$, but the converse is not true in general as it is shown by the following example.

Example 1. [17] Let $X = \{0,1,2\}$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(1,0) &= d(0,1) = d(2,1) = d(1,2) = 1, \\ d(0,2) &= d(2,0) = m, \end{aligned}$$

where, m is given real number such that $m \geq 2$. It is easy to check that for all $x, y, z \in X$

$$d(x,y) \leq \frac{m}{2}(d(x,z) + d(z,y)).$$

Therefore, (X,d) is a b-metric space with a coefficient $s = \frac{m}{2}$. The ordinary triangle inequality does not hold if $m > 2$ and so (X,d) is not a metric space.

Example 2. [13] Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

For other examples of a b-metric, see [1].

Definition 2. [18] Let (X,d) be a b-metric space and $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ is said to be

(i) Convergent to $x \in X$ if $\lim_{n\to+\infty} d(x_n, x) = 0$. In this case, we write $\lim_{n\to+\infty} x_n = x$.
(ii) A Cauchy sequence if $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$.
(iii) (X,d) is complete if every Cauchy sequence in X is convergent.

Remark 1. In general, a b-metric need not be continuous in each variable [13].

The following lemma was established by [12].

Lemma 1. *Let (X,d) be a b-metric space with a coefficient $s \geq 1$ and $\{x_n\}$ a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),\ n = 1, 2, ...,$$

where $0 \leq \lambda < 1$. Then $\{x_n\}$ is a Cauchy sequence.

Recently, Ansari [19] introduced the concept of following C-class functions which covers a large class of contractive conditions.

Definition 3. [19] A continuous function $F : [0,+\infty) \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,+\infty)$; the following conditions hold

(c1) $F(s,t) \leq s$,
(c2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 3. The following examples show that the class C is nonempty;

1. $F(s,t) = s - t$,
2. $F(s,t) = ms$; for some $m \in (0,1)$,
3. $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,1)$,
4. $F(s,t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$,
5. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$,
6. $F(s,t) = s\beta(s)$, $\beta : [0,+\infty) \to (0,1)$ is continuous,
7. $F(s,t) = s - \frac{t}{k+t}$.
8. $F(s,t) = s - (\frac{2+t}{1+t})t$,
9. $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.

Let Φ_u denote the class of the functions $\varphi : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions;

(a) φ is continuous,
(b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

In 1984, Khan *et al.*, [20] introduced altering distance function as follows;

Definition 4. [20] A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied;

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 5. A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0,+\infty)$, we have

$$x \le y \Rightarrow F(\psi(x), \varphi(x)) \le F(\psi(y), \varphi(y)).$$

Example 4. Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$ and $\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$, then (ψ, φ, F) is monotone.

2. Main result

In this section we assume ψ is altering distance function, φ is ultra altering distance function and F is a C-class function.

Theorem 1. *Let (X,d) be a b-complete metric space and T be a self-mapping on X that satisfies the following contractive condition;*

$$\psi\left(d(Tx,Ty)\right) \le F\left(\psi(M(x,y)), \varphi(M(x,y))\right), \tag{1}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in C$ such that (ψ, φ, F) is monotone and

$$M(x,y) = \max\left\{d(x,y), \frac{d^2(x,y)}{1+d(y,Ty)}, \frac{d^2(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Ty,Tx)}\right\}. \tag{2}$$

Then, T has a unique fixed point in X.

Proof. Let x in X and $\{x_n\}_n$ be a sequence in X defined as

$$Tx_n = x_{n+1}, y_n = x_{n-1} \; n = 0,1,2\ldots.$$

Applying the inequality (1), we obtain

$$\psi\left(d(Tx_n, Tx_{n-1})\right) \le F\left(\psi(M(x_n, x_{n-1})), \varphi(M(x_n, x_{n-1}))\right),$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{d(x_n, x_{n-1}), \frac{d^2(x_n, x_{n-1})}{1+d(x_{n-1}, Tx_{n-1})}, \frac{d^2(x_{n-1}, Tx_{n-1})}{1+d(x_n, x_{n-1})}, \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{1+d(Tx_{n-1}, Tx_n)}\right\} \\ &\le d(x_{n-1}, x_n). \end{aligned}$$

Thus

$$\begin{aligned} \psi\left(d(Tx_n, Tx_{n-1})\right) &\le F\left(\psi(d(x_n, x_{n-1})), \varphi(d(x_n, x_{n-1}))\right) \\ &\le \psi(d(x_n, x_{n-1})). \end{aligned}$$

Since ψ is non-decreasing, then $d(Tx_n, Tx_{n+1}) \le d(x_n, x_{n-1})$. This means $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Thus it converges and there exists $r \ge 0$ such that $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = r$. Taking $n \to +\infty$, then contractive condition implies$\psi(r) \le F(\psi(r), \varphi(r)) \le \psi(r)$. So, $\psi(r) = 0$. Therefore $r = 0$, that is $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can nd two sequences of positive integers $m(k)$ and $n(k)$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$. Let $n(k)$ be the smallest positive integer $n(k) > m(k) > k$, such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon,\ d(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon.$$

Then, we find $\psi(\varepsilon) = 0$ which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy sequence in X. Since (X, d) is a complete b-metric space, so there exists $u \in X$, such that $\lim_{n\to+\infty} x_n = u$.

Uniqueness of fixed point

Let $v \neq u$ be another fixed point of f, then from the contraction condition, we have

$$\psi\left(d(u,v)\right) \leq \psi\left(sd(u,v)\right) = \psi\left(sd(Tu,Tv)\right) \leq F\left(\psi(M(u,v)), \varphi(M(u,v))\right),$$

where

$$M(u,v) = \max\left\{d(u,v), \frac{d^2(u,v)}{1+d(v,Tv)}, \frac{d^2(v,Tv)}{1+d(u,v)}, \frac{d(u,Tu)d(v,Tv)}{1+d(Tv,Tu)}\right\}.$$

Then $\psi\left(d(u,v)\right) = 0$, thus $d(u,v) = 0$. This shows T has a unique fixed point. □

The following example supports our Theorem 1.

Example 5. Let the complete b-metric space (X, d) with $X = \left[0, \frac{1}{2}\right]$ and

$$d(x,y) = |x-y| \ \ for\ all\ \ x,y \in X.$$

Consider $T : X \to X$ be given by $Tx = \frac{x}{4}$ for all $x \in X$. Then, for $\psi\left(t\right) = t$ and $F(s,t) = ms$ for some $m \in (0,1)$, we have

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{4}\left|x-y\right| \leq \frac{1}{2}\left|x-y\right| \\ &\leq \frac{1}{2}d(x,y) \leq \frac{1}{2}M(x,y). \end{aligned}$$

Thus, T is satisfying all the conditions of Theorem 1 and 0 is its fixed point, which is unique.

The following results can be obtained immediately from Theorem 1.

Corollary 1. *Let (X,d) be a complete b-metric space and T be a self-mapping on X that satisfies the following contractive condition;*

$$\psi\left(d(Tx,Ty)\right) \leq \psi(M(x,y)) - \varphi(M(x,y)),$$

for all $x, y \in X$ where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ such that (ψ, φ, F) is monotone and

$$M(x,y) = \max\left\{d(x,y), \frac{d^2(x,y)}{1+d(y,Ty)}, \frac{d^2(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Ty,Tx)}\right\}.$$

Then, T has a unique fixed point in X.

Proof. Taking $F(s,t) = s - t$, in Theorem 1, we obtain the desired result. □

Corollary 2. *Let (X,d) be a complete b-metric space and T be a self-mapping on X that satisfies the following contractive condition;*

$$\psi\left(d(Tx,Ty)\right) \leq M(x,y)\Phi\left(\varphi(M(x,y)\right),$$

for all $x, y \in X$ *where* $\psi \in \Psi$; $\varphi \in \Phi_u$ *and* $F \in C$ *such that* (ψ, φ, F) *is monotone and*

$$M(x,y) = \max\left\{d(x,y), \frac{d^2(x,y)}{1+d(y,Ty)}, \frac{d^2(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Ty,Tx)}\right\}.$$

Then, T has a unique fixed point in X

Proof. Taking $\psi(t) = t$ and $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,1)$ in Theorem 1, we obtain the desired result. □

Corollary 3. *Let* (X, d) *be a complete b-metric space and T be a self-mapping on X that satisfies the following contractive condition;*

$$d(Tx, Ty) \leq \frac{M(x,y)}{(1+M(x,y))^r},$$

for all $x, y \in X$ *where* $\psi \in \Psi$; $\varphi \in \Phi_u$ *,*$r \in (0,1)$ *and* $F \in C$ *such that* (ψ, φ, F) *is monotone and*

$$M(x,y) = \max\left\{d(x,y), \frac{d^2(x,y)}{1+d(y,Ty)}, \frac{d^2(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Ty,Tx)}\right\}.$$

Then, T has a unique fixed point in X

Proof. Taking $F(s,t) = s\Phi(t)$, $(s, t > 0)$ in Theorem 1, we obtain the desired result. □

3. Application to integral equations

Let $X = C[a,b]$ be a set of all real valued continuous functions on $[a,b]$, where $[a,b]$ is closed and bounded interval in $\mathbb{R}$. For a real number $p > 1$, define $d : X \times X \to \mathbb{R}_+$ by

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|^p,$$

for all $x, y \in X$. Therefore, (X, d) is a complete b-metric space with $s = 2^{p-1}$. We apply Theorem 1 to establish the existence of solution of Fredholm type defined by

$$x(t) = f(t) + \lambda \int_a^b K(t,\tau,x)d\tau, \tag{3}$$

where $x \in C[a,b]$ is the unknown function, $\lambda \in \mathbb{R}, t, \tau \in [a,b]$, $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are given continuous functions.

Theorem 2. *We assume the following conditions;*

(i) *There exists a continuous function* $\psi : [a,b] \times [a,b] \to \mathbb{R}_+$ *such that for all* $x, y \in X$, $\lambda \in \mathbb{R}$ *and* $t, \tau \in [a,b]$, *we get*$|K(t,\tau,x) - K(t,\tau,y)|^p \leq \psi(t,\tau).\, |x-y|^p$,

(ii) $|\lambda| \leq 1$,

(iii) $\max_{t\in[a,b]} \int_a^b \psi(t,\tau)d\tau \leq \frac{1}{(b-a)^{p-1}}$, *where* $s = \frac{1}{2^{p-1}}$.

Then, the Equation (3) has a solution $z \in C[a,b]$.

Proof. Define the mapping $T : X \to X$ by

$$Tx(t) = f(t) + \lambda \int_a^b K(t,\tau,x(\tau))d\tau,$$

for all $t \in [a,b]$. So, the existence of a solution of (3) is equivalent to the existence of fixed point T. Let $q \in \mathbb{R}$ such that $\frac{1}{p}+\frac{1}{q}=1$. Using the Hölder inequality, and conditions (i)-(iii), we have

$$\begin{aligned}
d(Tx,Ty) &= \max_{t\in[a,b]} |Tx(t)-Ty(t)|^p \\
&\leq |\lambda|^p \max_{t\in[a,b]} \left(\int_a^b |K(t,\tau,x)-K(t,\tau,y|^p \, d\tau \right) \\
&\leq \left[\max_{t\in[a,b]} \left(\int_a^b 1^q dz \right)^{\frac{1}{q}} \left(\int_a^b |(K(t,\tau,x)-K(t,\tau,y))|^p \, d\tau \right)^{\frac{1}{p}} \right]^p \\
&\leq (b-a)^{\frac{p}{q}} \left[\max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) \, |x-y|^p \, d\tau \right) \right] \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau \right) d(x,y) \\
&\leq (b-a)^{p-1} \frac{1}{(b-a)^{p-1}} . M(x,y) .
\end{aligned}$$

Thus

$$d(Tx,Ty) \leq M(x,y) .$$

Hence, all the conditions of Theorem 1 hold. Consequently, the Equation (3) has a solution $z \in C[a,b]$. □

Acknowledgments: The author is grateful to the referees for the useful comments.

Conflicts of Interest: "The author declares no conflict of interest."

References

[1] Bakhtin, I. (1989). The contraction mapping principle in quasimetric spaces. *Func. An., Gos. Ped. Inst. Unianowsk, 30*, 26-37.

[2] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis, 1*(1), 5-11.

[3] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux quations intégrales. *Fundamenta Mathematicae, 3*(1), 133-181.

[4] Khamsi, M. A., & Hussain, N. (2010). KKM mappings in metric type spaces. *Nonlinear Analysis: Theory, Methods & Applications, 73*(9), 3123-3129.

[5] Akkouchi, M. (2011). Common fixed point theorems for two self-mappings of a b-metric space under an implicit relation. *Hacettepe Journal of Mathematics and Statistics, 40*(6), 805-810.

[6] Akkouchi, M. (2019). A common fixed point result for two pairs of weakly tangential maps in b-metric spaces. *Journal of International Mathematical Virtual Institute, 9*, 189-204.

[7] Babu, G. V. R., & Dula, T. M. (2018). Fixed points of almost generalized (α,β)-(ψ,φ) contractive mappings in b-metric spaces. *Facta Universitatis, Series Mathematics and Informatics, 33*(2), 177-196.

[8] Boriceanu, M. (2009). Fixed point theory for multivalued generalized contraction on a set with two b-metrics, *Studia Universitatis Babes-Bolyai, Series Mathematica, 54*(3), 3-14.

[9] Demma, M., & Vetro, P. (2015). Picard sequence and fixed point results on b-metric spaces. *Journal of Function Spaces, 2015*, Article ID 189861.

[10] Hamaizia, H., & Aliouche, A. (2017). A fixed point theorem in b-metric spaces and its application in linear integral equation. *Asian Journal of Mathematics and Computer Research, 15*(1) , 23-29.

[11] Hieu, N. T., & Dung, N. V. (2015). Some fixed point results for generalized rational type contraction mappings in partially ordered b-metric spaces. *Facta Universitatis, Series: Mathematics and Informatics, 30*, 49-66.

[12] Huang, H., Deng, G., & Radenovic, S. (2018). Fixed point theorems in $b-$metric spaces with applications to differential equations. *Journal of Fixed Point Theory and Applications, 20*, Article No 52.

[13] Hussain, N., Parvaneh, V., Roshan, J. R., & Kadelburg, Z. (2013). Fixed points of cyclic (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications. *Fixed Point Theory and Applications, 2013*, Article No 256.

[14] Rao, K. P. R., & Rao, K. R. K. (2013). A common fixed point theorem for two hybrid pairs of mappings in $b-$metric spaces. *International Journal of Analysis, 2013*, Article ID 404838.

[15] Suzuki, T. (2017). Basic inequality on ab-metric space and its applications. *Journal of Inequalities and Applications, 2017*, Article No 256.

[16] Suzuki, T. (2018). Fixed point theorems for single-and set-valued F-contractions in b-metric spaces. *Journal of Fixed Point Theory and Applications*, *20*, Article No 35.

[17] Sintunavarat, W., Plubtieng, S., & Katchang, P. (2013). Fixed point result and applications on b-metric space endowed with an arbitrary binary relation. *Fixed Point Theory and Applications, 2013*, Article No 296.

[18] Czerwik, S. (1998). Nonlinear set-valued contraction mappings in $b-$metric spaces. *Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia, 46*, 263-276.

[19] Ansari, A. H. (2014). Note on φ-ψ-contractive type mappings and related fixed point. *The 2nd regional conference on mathematics and applications, Payame Noor University*, 377-380.

[20] Khan, M. S., Swaleh, M., & Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society, 30*(1), 1-9.