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Direct product of fuzzy multigroups under t-norms

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Abstract: This paper proposes the concept of direct product of fuzzy multigroups under t-norms and some of their basic properties are obtained. Next, we investigate and obtain some new results of strong upper alpha-cut, weak upper alpha-cut, strong lower alpha-cut and weak lower alpha-cut of them. Later, we prove conjugation and commutation between them. Finally, the notion of homomorphism in the context of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups under t-norms in terms of homomorphic images and homomorphic preimages, respectively, were presented.

Keywords: Fuzzy multigroups, t-norm, direct products, homomorphisms.

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1. Introduction

Theory of multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. The term multiset (mset in short) as Knuth [1] notes, was first suggested by De Bruijn [2] in a private communication to him. The concept of fuzzy sets proposed by Zadeh [3] is a mathematical tool for representing vague concepts. The idea of fuzzy multisets was conceived by Yager [4] as the generalization of fuzzy sets in multisets framework.

The concept of fuzzy multigroups was introduced by Shinoj *et al.*, [5] as an application of fuzzy multisets to group theory, and some properties of fuzzy multigroups were presented. Ejegwa introduced the concept of fuzzy multigroupoids and presented the idea of fuzzy submultigroups with a number of results and more properties of abelian fuzzy multigroups were explicated [6–8]. Also Ejegwa introduced direct product in fuzzy multigroup setting as an extension of direct product of fuzzy subgroups [9]. In mathematics, a t-norm (also T-norm or, unabbreviated, triangular norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. A t-norm generalizes intersection in a lattice and conjunction in logic. The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t-norms are used to generalize triangle inequality of ordinary metric spaces. The author by using norms, investigated some properties of fuzzy algebraic structures [10–15]. The author [15] defined fuzzy multigroups under t-norms and some properties of them are explored and some related results are obtained.

In this paper, we introduce the concept of direct product of fuzzy multigroups under t-norms $(TFSM(G))$ and investigate some properties and results about them. We prove that direct products of $TFSM(G)$ are also $TFSM(G)$. Next we investigate and obtain some new results of strong upper alpha-cut, weak upper alpha-cut, strong lower alpha-cut and weak lower alpha-cut of direct product of fuzzy Multigroups under t-norms. Later we prove that if $A, C \in TFMS(G)$ and $B, D \in TFMS(H)$ such that A is conjugate to B and C is conjugate to D, then $A \times C$ is conjugate to $B \times D$. Also A and B are commutative if and only if $A \times B$ is a commutative. Finally, we define group homomorphisms on direct propduct of fuzzy multigroups under t-norms and we prove that image and pre image of direct propduct of fuzzy multigroups under t-norms is also fuzzy multigroups under t-norms.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [15–24].

Definition 1. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$.

Definition 2. Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \to [0,1]$$

of which the value is a multiset of the unit interval $I = [0,1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n, ...\} \forall x \in X,$$

where $\mu^1, \mu^2, ..., \mu^n, ... \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n \geq ...).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\},$$

where $\mu^1, \mu^2, ..., \mu^n \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n),$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0,1]$ and $i = 1, 2, ..., n$.

Now, a fuzzy multiset A is given as

$$A = \left\{ \frac{CM_A(x)}{x} : x \in X \right\} \text{ or } A = \{(CM_A(x), x) : x \in X\}.$$

The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 1. Consider the set $X = \{a, b, c\}$. Then for $CM_A(a) = \{1, 0.5, 0.4\}$, $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$ we get that A is a fuzzy multiset of X written as $A = \left\{ \frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6}{b} \right\}$.

Definition 3. Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 4. Let $A \in FMS(X)$ and $\alpha \in [0,1]$. Then we define the following notions:

1. $A_\star = \{x \in X \mid CM_A(x) > 0\}$.
2. $A^\star = \{x \in X \mid CM_A(x) = CM_A(e_X)\}$ where e_X is the identity element of X.
3. $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ is called strong upper alpha-cut of A.
4. $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$ is called weak upper alpha-cut of A.
5. $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ is called strong lower alpha-cut of A.
6. $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$ is called weak lower alpha-cut of A.

Definition 5. Let $A, B \in FMG(X)$. We say that A is conjugate to B if for all $x, y \in X$ we have $CM_A(x) = CM_B\left(yxy^{-1}\right)$.

Definition 6. Let $A \in FMG(X)$. We say that A is commutative if $CM_A(xy) = CM_A(yx)$ for all $x, y \in X$.

Definition 7. A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),

(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),
for all $x,y,z \in [0,1]$.

We say that T be idempotent if $T(x,x) = x$ for all $x \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 2. (1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) Algebraic product t-norm $T_p(x,y) = xy$.
(4) Drastic T-norm $T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$
(5) Nilpotent minimum t-norm $T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$
(6) Hamacher product t-norm $T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$.

Lemma 1. *Let T be a t-norm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 8. Let $A \in FMS(G)$. Then A is said to be a fuzzy multigroup of G under t-norm T if it satisfies the following two conditions:
(1) $CM_A(xy) \geq T(CM_A(x), CM_A(y))$,
(2) $CM_A(x^{-1}) \geq CM_A(x)$,
for all $x,y \in G$.

The set of all fuzzy multisets of G under t-norm T is depicted by $TFMS(G)$.

Theorem 1. *Let $A \in TFMS(G)$. If T be idempotent, then for all $x \in G$, we have and $n \geq 1$,*
(1) $CM_A(e) \geq CM_A(x)$;
(2) $CM_A(x^n) \geq CM_A(x)$;
(3) $CM_A(x) = CM_A(x^{-1})$.

3. Direct product of fuzzy multigroups under t-norms

Definition 9. Let $A \in TFMS(G)$ and $B \in TFMS(H)$. The direct product of A and B, denoted by $A \times B$, is characterized by a count membership function

$$CM_{A\times B} : G \times H \to [0,1]$$

such that

$$CM_{A\times B}(x,y) = T(CM_A(x), CM_B(y))$$

for all $x \in G$ and $y \in H$.

Example 3. Let $G = \{1, x\}$ be a group, where $x^2 = 1$ and $H = \{e,a,b,c\}$ be a Klein 4-group, where $a^2 = b^2 = c^2 = e$. Suppose

$$A = \left\{ \frac{0.9, 0.8}{1}, \frac{0.7, 0.6}{x} \right\}$$

and

$$B = \left\{ \frac{1, 0.85}{e}, \frac{0.35, 0.25}{a}, \frac{0.10, 0.50}{b}, \frac{0.8, 0.6}{c} \right\}$$

be fuzzy multigroups of G and H. Let

$$G \times H = \{(1, e), (1, a), (1, b), (1, c), (x, e), (x, a), (x, b), (x, c)\}$$

be a a group from the classical sense. Define

$$A \times B = \left\{ \frac{0.9, 0.8}{(1, e)}, \frac{0.35, 0.25}{(1, a)}, \frac{0.10, 0.50}{(1, b)}, \frac{0.8, 0.6}{(1, c)}, \frac{0.7, 0.6}{(x, e)}, \frac{0.35, 0.25}{(x, a)}, \frac{0.10, 0.50}{(x, b)}, \frac{0.7, 0.6}{(x, c)} \right\}$$

and let $T_m(x, y) = \min\{x, y\}$ be a standard intersection t-norm for all $x, y \in [0, 1]$. Then

$$A \times B \in TFMS(G \times H).$$

Proposition 1. *Let $A_i \in TFMS(G_i)$ for $i = 1, 2$. Then $A_1 \times A_2 \in TFMS(G_1 \times G_2)$.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in G_1 \times G_2$. Then

$$\begin{aligned} (CM_{A\times B})((a_1, b_1)(a_2, b_2)) &= (CM_{A\times B})(a_1a_2, b_1b_2) = T(CM_A(a_1a_2), CM_B(b_1b_2)) \\ &\geq T(T(CM_A(a_1), CM_A(a_2)), T(CM_B(b_1), CM_B(b_2))) \\ &= T(T(CM_A(a_1), CM_B(b_1), T(CM_A(a_2), CM_B(b_2)) \\ &= T((CM_{A\times B})(a_1, b_1), (CM_{A\times B})(a_2, b_2)). \end{aligned}$$

Also

$$\begin{aligned} (CM_{A\times B})(a_1, b_1)^{-1} &= (CM_{A\times B})(a_1^{-1}, b_1^{-1}) \\ &= T(CM_A(a_1^{-1}), CM_B(b_1^{-1})) \geq T(CM_A(a_1), CM_B(b_1)). \end{aligned}$$

Thus $A_1 \times A_2 \in TFMS(G_1 \times G_2)$. □

Corollary 1. *Let $A \in TFMS(G)$ and $B \in TFMS(H)$. Then*

$$A \times 1_H, 1_G \times B \in TFMS(G \times H).$$

Corollary 2. *Let $A_i \in TFMS(G_i)$ for $i = 1, 2, ..., n$. Then*

$$A_1 \times A_2 \times ... \times A_n \in TFMS(G_1 \times G_2 \times ... \times G_n).$$

Proposition 2. *Let $A \in TFMS(G)$ and $B \in TFMS(H)$ such that T be idempotent t-norm. Then for all $\alpha \in [0, 1]$ the following assertions hold:*

(1) $(A \times B)_\star = A_\star \times B_\star$.
(2) $(A \times B)^\star = A^\star \times B^\star$.
(3) $(A \times B)_{[\alpha]} = A_{[\alpha]} \times B_{[\alpha]}$.
(4) $(A \times B)_{(\alpha)} = A_{(\alpha)} \times B_{(\alpha)}$.
(5) $(A \times B)^{[\alpha]} = A^{[\alpha]} \times B^{[\alpha]}$.
(6) $(A \times B)^{(\alpha)} = A^{(\alpha)} \times B^{(\alpha)}$.

Proof. (1) We know that $(A\times B)_\star=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)>0\}$. Then $(x,y)\in(A\times B)_\star \Leftarrow CM_{A\times B}(x,y)>0 \Leftarrow T(CM_A(x),CM_B(y))>0 \Leftarrow T(CM_A(x),CM_B(y))>0=T(0,0)\Leftarrow CM_A(x)>0$ *and* $CM_B(y)>0 \Leftarrow x\in A_\star$ *and* $y\in B_\star \Leftarrow (x,y)\in A_\star\times B_\star$. Hence $(A\times B)_\star=A_\star\times B_\star$.

(2) As $(A\times B)^\star=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)=CM_{A\times B}(e_G,e_H)\}$ so $(x,y)\in(A\times B)^\star \Leftarrow CM_{A\times B}(x,y)=CM_{A\times B}(e_G,e_H)\Leftarrow T(CM_A(x),CM_B(y))=T(CM_A(e_G),CM_B(e_H))\Leftarrow CM_A(x)=CM_A(e_G)$ *and* $CM_B(y)=CM_B(e_H)\Leftarrow x\in A^\star$ *and* $y\in B^\star \Leftarrow (x,y)\in A^\star\times B^\star$. Thus $(A\times B)^\star=A^\star\times B^\star$.

(3) Let $(A\times B)_{[\alpha]}=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)\geq\alpha\}$. Now $(x,y)\in(A\times B)_{[\alpha]}\Leftarrow CM_{A\times B}(x,y)\geq\alpha\Leftarrow T(CM_A(x),CM_B(y))\geq\alpha\Leftarrow T(CM_A(x),CM_B(y))\geq\alpha=T(\alpha,\alpha)\Leftarrow CM_A(x)\geq\alpha$ *and* $CM_B(y)\geq\alpha\Leftarrow x\in A_{[\alpha]}$ *and* $y\in B_{[\alpha]}\Leftarrow (x,y)\in A_{[\alpha]}\times B_{[\alpha]}$. Thus $(A\times B)_{[\alpha]}=A_{[\alpha]}\times B_{[\alpha]}$.

(4) Since $(A\times B)_{(\alpha)}=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)>\alpha\}$, so $(x,y)\in(A\times B)_{(\alpha)}\Leftarrow CM_{A\times B}(x,y)>\alpha\Leftarrow T(CM_A(x),CM_B(y))>\alpha\Leftarrow T(CM_A(x),CM_B(y))>\alpha=T(\alpha,\alpha)\Leftarrow CM_A(x)>\alpha$ *and* $CM_B(y)>\alpha\Leftarrow x\in A_{(\alpha)}$ *and* $y\in B_{(\alpha)}\Leftarrow (x,y)\in A_{(\alpha)}\times B_{(\alpha)}$. So $(A\times B)_{(\alpha)}=A_{(\alpha)}\times B_{(\alpha)}$.

(5) Because $(A\times B)^{[\alpha]}=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)\leq\alpha\}$, then $(x,y)\in(A\times B)^{[\alpha]}\Leftarrow CM_{A\times B}(x,y)\leq\alpha\Leftarrow T(CM_A(x),CM_B(y))\leq\alpha(x,y)\in(A\times B)^{[\alpha]}\Leftarrow CM_{A\times B}(x,y)\leq\alpha\Leftarrow T(CM_A(x),CM_B(y))\leq\alpha\Leftarrow T(CM_A(x),CM_B(y))\leq\alpha=T(\alpha,\alpha)\Leftarrow CM_A(x)\leq\alpha$ *and* $CM_B(y)\leq\alpha\Leftarrow T(CM_A(x),CM_B(y))\leq\alpha=T(\alpha,\alpha)\Leftarrow CM_A(x)\leq\alpha$ *and* $CM_B(y)\leq\alpha\Leftarrow x\in A^{[\alpha]}$ *and* $y\in B^{[\alpha]}\Leftarrow (x,y)\in A^{[\alpha]}\times B^{[\alpha]}$. Therefore $(A\times B)^{[\alpha]}=A^{[\alpha]}\times B^{[\alpha]}$.

(6) Because of $(A\times B)^{(\alpha)}=\{(x,y)\in G\times H \mid CM_{A\times B}(x,y)<\alpha\}$, then $(x,y)\in(A\times B)^{(\alpha)}\Leftarrow CM_{A\times B}(x,y)<\alpha\Leftarrow T(CM_A(x),CM_B(y))<\alpha\Leftarrow T(CM_A(x),CM_B(y))<\alpha=T(\alpha,\alpha)\Leftarrow CM_A(x)<\alpha$ *and* $CM_B(y)<\alpha\Leftarrow x\in A^{(\alpha)}$ *and* $y\in B^{(\alpha)}\Leftarrow (x,y)\in A^{(\alpha)}\times B^{(\alpha)}$. Hence $(A\times B)^{(\alpha)}=A^{(\alpha)}\times B^{(\alpha)}$. □

Proposition 3. *Let $A\in TFMS(G)$ and $B\in TFMS(H)$ such that T be idempotent t-norm. Then for all $(x,y)\in G\times H$ the following assertions hold:*

(1) $CM_{A\times B}(e_G,e_H)\geq CM_{A\times B}(x,y)$,

(2) $CM_{A\times B}((x,y)^n)\geq CM_{A\times B}(x,y)$,

(3) $CM_{A\times B}(x,y)=CM_{A\times B}(x^{-1},y^{-1})$.

Proof. Using Proposition 1 we get that $A\times B\in TFMS(G\times H)$. Now Theorem 1 gives us that assertions are hold. □

Proposition 4. *Let $A\in TFMS(G)$ and $B\in TFMS(H)$ such that T be idempotent t-norm. Then for all $\alpha\in[0,1]$ the following assertions hold:*

(1) $(A\times B)_\star$ is a subgroup of $G\times H$,

(2) $(A\times B)^\star$ is a subgroup of $G\times H$,

(3) $(A\times B)_{[\alpha]}$ is a subgroup of $G\times H$,

(4) $(A\times B)_{(\alpha)}$ is a subgroup of $G\times H$.

Proof. (1) Let $(x_1,y_1),(x_2,y_2)\in(A\times B)_\star$. We need to prove that $(x_1,y_1)(x_2,y_2)^{-1}\in(A\times B)_\star$.
As $(x_1,y_1),(x_2,y_2)\in(A\times B)_\star$, so $CM_{A\times B}(x_1,y_1)>0$ and $CM_{A\times B}(x_2,y_2)>0$.
Now

$$\begin{aligned}
CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\times B}(x_1x_2^{-1},y_1y_2^{-1})=T(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\geq T(T(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\
&\geq T(T(CM_A(x_1),CM_A(x_2)),T(CM_B(y_1),CM_B(y_2)))\\
&= T(T(CM_A(x_1),CM_B(y_1)),T(CM_A(x_2),CM_B(y_2)))\\
&= T(CM_{A\times B}(x_1,y_1),CM_{A\times B}(x_2,y_2))>T(0,0)=0\,.
\end{aligned}$$

Thus $CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1})>0$, which means that $(x_1,y_1)(x_2,y_2)^{-1}\in(A\times B)_\star$. Hence $(A\times B)_\star$ is a subgroup of $G\times H$.

(2) Let $(x_1,y_1),(x_2,y_2)\in(A\times B)_\star$. We need to prove that $(x_1,y_1)(x_2,y_2)^{-1}\in(A\times B)^\star$.

Because $(x_1,y_1),(x_2,y_2) \in (A\times B)^\star$ then $CM_{A\times B}(x_1,y_1) = CM_{A\times B}(x_2,y_2) = CM_{A\times B}(e_G,e_H)$, which means that $T(CM_A(x_1),CM_B(y_1)) = T(CM_A(x_2),CM_B(y_2)) = T(CM_A(e_G),CM_B(e_H))$, so $CM_A(x_1) = CM_A(x_2) = CM_A(e_G)$ and $CM_A(y_1) = CM_A(y_2) = CM_A(e_H)$. Thus

$$\begin{aligned}
CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\times B}(x_1x_2^{-1},y_1y_2^{-1}) = T(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\geq T(T(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\
&\geq T(T(CM_A(x_1),CM_A(x_2)),T(CM_B(y_1),CM_B(y_2)))\\
&= T(T(CM_A(e_G),CM_A(e_G)),T(CM_B(e_H),CM_B(e_H)))\\
&= T(CM_A(e_G),CM_B(e_H)) = CM_{A\times B}(e_G,e_H)\\
&\geq CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1})\,. \textit{(Proposition 2(1))}
\end{aligned}$$

Thus $CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) = CM_{A\times B}(e_G,e_H)$, so $(x_1,y_1)(x_2,y_2)^{-1} \in (A\times B)^\star$. Hence we obtain that $(A\times B)^\star$ is a subgroup of $G\times H$.

(3) Let $(x_1,y_1),(x_2,y_2) \in (A\times B)_{[\alpha]}$. We need to show that $(x_1,y_1)(x_2,y_2)^{-1} \in (A\times B)_{[\alpha]}$.
As $(x_1,y_1),(x_2,y_2) \in (A\times B)_{[\alpha]}$ so $CM_{A\times B}(x_1,y_1) \geq \alpha$ and $CM_{A\times B}(x_2,y_2) \geq \alpha$.
Now

$$\begin{aligned}
CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\times B}(x_1x_2^{-1},y_1y_2^{-1}) = T(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\geq T(T(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\
&\geq T(T(CM_A(x_1),CM_A(x_2)),T(CM_B(y_1),CM_B(y_2)))\\
&= T(T(CM_A(x_1),CM_B(y_1)),T(CM_A(x_2),CM_B(y_2)))\\
&= T(CM_{A\times B}(x_1,y_1),CM_{A\times B}(x_2,y_2)) \geq T(\alpha,\alpha) = \alpha\,.
\end{aligned}$$

Thus $CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) \geq \alpha$ which means that $(x_1,y_1)(x_2,y_2)^{-1} \in (A\times B)_{[\alpha]}$. Hence $(A\times B)_{[\alpha]}$ is a subgroup of $G\times H$.

(4) If $(x_1,y_1),(x_2,y_2) \in (A\times B)_{(\alpha)}$, then $CM_{A\times B}(x_1,y_1) > \alpha$ and $CM_{A\times B}(x_2,y_2) > \alpha$. Now

$$\begin{aligned}
CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\times B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\times B}(x_1x_2^{-1},y_1y_2^{-1}) = T(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\geq T(T(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\
&\geq T(T(CM_A(x_1),CM_A(x_2)),T(CM_B(y_1),CM_B(y_2)))\\
&= T(T(CM_A(x_1),CM_B(y_1)),T(CM_A(x_2),CM_B(y_2)))\\
&= T(CM_{A\times B}(x_1,y_1),CM_{A\times B}(x_2,y_2)) > T(\alpha,\alpha) = \alpha\,.
\end{aligned}$$

Thus $CM_{A\times B}((x_1,y_1)(x_2,y_2)^{-1}) > \alpha$ which means that $(x_1,y_1)(x_2,y_2)^{-1} \in (A\times B)_{(\alpha)}$. Hence $(A\times B)_{(\alpha)}$ is a subgroup of $G\times H$. □

Proposition 5. *Let $A \in TFMS(G)$ and $B \in TFMS(H)$. If $A\times B \in TFMS(G\times H)$, then at least one of the following statements hold:*
(1) $CM_B(e_H)) \geq CM_A(x)$ for all $x \in G$,
2) $CM_A(e_G)) \geq CM_B(y)$ for all $y \in G$.

Proof. Suppose that none of the statements holds, then we can find $a \in G$ and $b \in H$ such that $CM_A(a) > CM_B(e_H)$ and $CM_B(b) > CM_A(e_G)$. Now

$$\begin{aligned}
CM_{A\times B}(a,b) &= T(CM_A(a),CM_B(b))\\
&> T(CM_B(e_H),CM_A(e_G))\\
&= T(CM_A(e_G),CM_B(e_H)) = CM_{A\times B}(e_G,e_H)\,.
\end{aligned}$$

Thus $CM_{A\times B}(a,b) > CM_{A\times B}(e_G, e_H)$, which is contradiction with Proposition 2(1), hence at least one of the statements hold. □

Proposition 6. *Let $A \in FMS(G)$ and $B \in FMS(H)$ such that $A \times B \in TFMS(G \times H)$ and $CM_A(x) \le CM_B(e_H)$ for all $x \in G$. Then $A \in TFMS(G)$.*

Proof. As $CM_A(x) \le CM_B(e_H)$ for all $x \in G$, so $CM_A(y) \le CM_B(e_H)$ and $CM_A(xy) \le CM_B(e_H) = CM_B(e_H e_H)$ for all $y \in G$. Then

$$\begin{aligned} CM_A(xy) &= T(CM_A(xy), CM_B(e_H e_H)) \\ &= CM_{A\times B}(xy, e_H e_H) \\ &= CM_{A\times B}((x, e_H)(y, e_H)) \\ &\ge T(CM_{A\times B}(x, e_H), CM_{A\times B}(y, e_H)) \\ &= T(T(CM_A(x), CM_B(e_H)), T(CM_A(y), CM_B(e_H))) \\ &= T(CM_A(x), CM_A(y))\,. \end{aligned}$$

Thus

$$CM_A(xy) \ge T(CM_A(x), CM_A(y)).$$

Also since $CM_A(x) \le CM_B(e_H)$ for all $x \in G$ so $CM_A(x^{-1}) \le CM_B(e_H)$. Thus

$$\begin{aligned} CM_A(x^{-1}) &= T(CM_A(x^{-1}), CM_A(e_H)) \\ &= T(CM_A(x^{-1}), CM_A(e_H^{-1})) \\ &= CM_{A\times B}((x, e_H)^{-1}) \\ &\ge CM_{A\times B}(x, e_H) \\ &= T(CM_A(x), CM_A(e_H)) = CM_A(x) \end{aligned}$$

and then $CM_A(x^{-1}) \ge CM_A(x)$. Therefore $A \in TFMS(G)$. □

Proposition 7. *Let $A \in FMS(G)$ and $B \in FMS(H)$ such that $A \times B \in TFMS(G \times H)$ and $CM_B(x) \le CM_A(e_G)$ for all $x \in H$. Then $B \in TFMS(H)$.*

Proof. The proof is similar to Proposition 6. □

Corollary 3. *Let $A \in FMS(G)$ and $B \in FMS(H)$ such that $A \times B \in TFMS(G \times H)$. Then either $A \in TFMS(G)$ or $B \in TFMS(H)$.*

Proof. Using Proposition 5 we get that $CM_B(e_H)) \ge CM_A(x)$ for all $x \in G$ or $CM_A(e_G)) \ge CM_B(y)$ for all $y \in G$. Then from Proposition 6 and Proposition 7 we have that either $A \in TFMS(G)$ or $B \in TFMS(H)$. □

Proposition 8. *Let $A, C \in TFMS(G)$ and $B, D \in TFMS(H)$. If A is conjugate to B and C is conjugate to D, then $A \times C$ is conjugate to $B \times D$.*

Proof. As A is conjugate to B so $CM_A(x) = CM_C(gxg^{-1})$ and as B is conjugate to D so $CM_B(y) = CM_D(hyh^{-1})$ for all $x, g \in G$ and $y, h \in H$. Now

$$\begin{aligned} CM_{A\times B}(x, y) &= T(CM_A(x), CM_B(y)) \\ &= T(CM_C(gxg^{-1}), CM_D(hyh^{-1})) \\ &= CM_{C\times D}(gxg^{-1}, hyh^{-1}) \\ &= CM_{C\times D}((g, h)(x, y)(g^{-1}, h^{-1})) \\ &= CM_{C\times D}((g, h)(x, y)(g, h)^{-1})\,. \end{aligned}$$

Thus $CM_{A\times B}(x,y) = CM_{C\times D}((g,h)(x,y)(g,h)^{-1})$ which means that $A\times C$ is conjugate to $B\times D$. $\square$

Proposition 9. *Let $A\in TFMS(G)$ and $B\in TFMS(H)$. Then A and B are commutative if and only if $A\times B$ is a commutative.*

Proof. Let $x_1,y_1\in G$ and $x_2,y_2\in H$ such that $x=(x_1,x_2)\in G\times H$ and $y=(y_1,y_2)\in G\times H$. Let A and B are commutative then $CM_A(x_1y_1)=CM_A(y_1x_1)$ and $CM_B(x_2y_2)=CM_B(y_2x_2)$. Which implies

$$\begin{aligned} CM_{A\times B}(xy) &= CM_{A\times B}((x_1,x_2)(y_1,y_2)) \\ &= CM_{A\times B}(x_1y_1,x_2y_2) \\ &= T(CM_A(x_1y_1),CM_B(x_2y_2)) \\ &= T(CM_A(y_1x_1),CM_B(y_2x_2)) \\ &= CM_{A\times B}(y_1x_1,y_2x_2) \\ &= CM_{A\times B}((y_1,y_2)(x_1,x_2)) \\ &= CM_{A\times B}(yx)\,. \end{aligned}$$

Thus $CM_{A\times B}(xy)=CM_{A\times B}(yx)$ and then $A\times B$ is a commutative.

Conversely, suppose that $A\times B$ is a commutative. Then $CM_{A\times B}(xy) = CM_{A\times B}(yx) \Leftarrow CM_{A\times B}((x_1,x_2)(y_1,y_2)) = CM_{A\times B}((y_1,y_2)(x_1,x_2)) \Leftarrow CM_{A\times B}(x_1y_1,x_2y_2) = CM_{A\times B}(y_1x_1,y_2x_2) \Leftarrow T(CM_A(x_1y_1),CM_B(x_2y_2)) = T(CM_A(y_1x_1),CM_B(y_2x_2)) \Leftarrow CM_A(x_1y_1) = CM_A(y_1x_1)$ and $CM_B(x_2y_2) = CM_B(y_2x_2)$ which gives us that A and B are commutative. $\square$

Definition 10. Let $G\times H$ and $I\times J$ be groups and $f: G\times H\to I\times J$ be a homomorphism. Let $A\times B\in FMS(G\times H)$ and $C\times D\in FMS(I\times J)$. Define $f(A\times B)\in FMS(I\times J)$ and $f^{-1}(C\times D)\in FMS(G\times H)$ as:

$$f(CM_{A\times B})(i,j) = (CM_{f(A\times B)})(i,j) = \begin{cases} \sup\{CM_{A\times B}(g,h)\mid g\in G, h\in H, f(g,h)=(i,j)\} & \text{if } f^{-1}(i,j)\neq\emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(CM_{C\times D}(g,h)) = CM_{f^{-1}(C\times D)}(g,h) = CM_{C\times D}(f(g,h))$$

for all $(g,h)\in G\times H$.

Proposition 10. *Let $G\times H$ and $I\times J$ be groups and $f: G\times H\to I\times J$ be an epimorphism. If $A\in TFMS(G), B\in TFMS(H)$ and $A\times B\in TFMS(G\times H)$, then $f(A\times B)\in TFMS(I\times J)$.*

Proof. (1) Let $X=(i_1,j_1)\in I\times J$ and $Y=(i_2,j_2)\in I\times J$ such that

$$f^{-1}(XY) = f^{-1}((i_1,j_1)(i_2,j_2)) = f^{-1}(i_1i_2,j_1j_2)\neq\emptyset.$$

Then

$$\begin{aligned} f(A\times B)(XY) &= f(A\times B)((i_1,j_1)(i_2,j_2)) \\ &= f(A\times B)(i_1i_2,j_1j_2) \\ &= \sup\{CM_{A\times B}(g_1g_2,h_1h_2)\mid g_1,g_2\in G, h_1,h_2\in H, f(g_1g_2,h_1h_2)=(i_1i_2,j_1j_2)\} \\ &= \sup\{CM_{A\times B}(g_1g_2,h_1h_2)\mid g_1,g_2\in G, h_1,h_2\in H, (f(g_1g_2),f(h_1h_2))=(i_1i_2,j_1j_2)\} \\ &= \sup\{CM_{A\times B}(g_1g_2,h_1h_2)\mid g_1,g_2\in G, h_1,h_2\in H, f(g_1g_2)=i_1i_2, f(h_1h_2)=j_1j_2\} \\ &= \sup\{T(CM_A(g_1g_2),CM_B(h_1h_2))\mid g_1,g_2\in G, h_1,h_2\in H, f(g_1g_2)=i_1i_2, f(h_1h_2)=j_1j_2\} \\ &= \sup\{T(CM_A(g_1g_2),CM_B(h_1h_2))\mid g_1,g_2\in G, h_1,h_2\in H, f(g_1g_2)=i_1i_2, f(h_1h_2)=j_1j_2\} \\ &\geq \sup\{T(T(CM_A(g_1),CM_A(g_2)),T(CM_B(h_1),CM_B(h_2)))\mid f(g_1g_2)=i_1i_2, f(h_1h_2)=j_1j_2\} \\ &= \sup\{T(T(CM_A(g_1),CM_B(h_1)),T(CM_A(g_2),CM_B(h_2)))\mid f(g_1g_2)=i_1i_2, f(h_1h_2)=j_1j_2\} \end{aligned}$$

$$
\begin{aligned}
&= \sup\{T(T(CM_A(g_1), CM_B(h_1)), T(CM_A(g_2), CM_B(h_2))) \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= \sup\{T(CM_{A\times B}(g_1, h_1), CM_{A\times B}(g_2, h_2)) \mid f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\} \\
&= T(\sup\{CM_{A\times B}(g_1, h_1) \mid f(g_1, h_1) = (i_1, j_1)\}, \sup\{CM_{A\times B}(g_2, h_2) \mid f(g_2, h_2) = (i_2, j_2)\}) \\
&= T(f(A \times B)(i_1, j_1), f(A \times B)(i_2, j_2)) = T(f(A \times B)(X), f(A \times B)(Y)).
\end{aligned}
$$

Thus

$$f(A \times B)(XY) \geq T(f(A \times B)(X), f(A \times B)(Y)).$$

(2) Let $X = (i, j) \in I \times J$ then

$$
\begin{aligned}
f(A \times B)(X^{-1}) &= f(A \times B)((i, j)^{-1}) \\
&= f(A \times B)(i^{-1}, j^{-1}) \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}, h^{-1}) = (i^{-1}, j^{-1})\} \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, (f(g^{-1}), f(h^{-1})) = (i^{-1}, j^{-1})\} \\
&= \sup\{CM_{A\times B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\} \\
&= \sup\{T(CM_A(g^{-1}), CM_B(h^{-1})) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\} \\
&\geq \sup\{T(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f^{-1}(g) = i^{-1}, f^{-1}(h) = j^{-1}\} \\
&= \sup\{T(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f(g) = i, f(h) = j\} \\
&= \sup\{CM_{A\times B}(g, h) \mid (g, h) \in G \times H, f(g, h) = (i, j)\} \\
&= f(A \times B)(i, j) = f(A \times B)(X)
\end{aligned}
$$

and then $f(A \times B)(X^{-1}) \geq f(A \times B)(X)$. Therefore $f(A \times B) \in TFMS(I \times J)$. □

Proposition 11. *Let $G \times H$ and $I \times J$ be groups and $f : G \times H \to I \times J$ be a homomorphism. If $C \in TFMS(I)$, $D \in TFMS(J)$ and $C \times D \in TFMS(I \times J)$, then $f^{-1}(C \times D) \in TFMS(G \times H)$.*

Proof. (1) Let $X = (g_1, h_1) \in G \times H$ and $Y = (g_2, h_2) \in G \times H$. Then

$$
\begin{aligned}
f^{-1}(CM_{C\times D})(XY) &= f^{-1}(CM_{C\times D})((g_1, h_1)(g_2, h_2)) \\
&= f^{-1}((CM_{C\times D})(g_1g_2, h_1h_2)) \\
&= CM_{C\times D}(f(g_1g_2, h_1h_2)) \\
&= CM_{C\times D}(f(g_1g_2), f(h_1h_2)) \\
&= T(CM_C(f(g_1g_2)), CM_D(f(h_1h_2))) \\
&= T(CM_C(f(g_1)f(g_2)), CM_D(f(h_1)f(h_2))) \\
&\geq T(T(CM_C(f(g_1)), CM_C(f(g_2))), T(CM_D(f(h_1)), CM_D(f(h_2))) \\
&= T(T(CM_C(f(g_1)), CM_D(f(h_1))), T(CM_C(f(g_2), CM_D(f(h_2))) \\
&= T(CM_{C\times D}(f(g_1), f(h_1)), CM_{C\times D}(f(g_2), f(h_2))) \\
&= T(CM_{C\times D}(f(g_1, h_1)), CM_{C\times D}(f(g_2, h_2))) \\
&= T(f^{-1}(CM_{C\times D})(g_1, h_1), f^{-1}(CM_{C\times D})(g_2, h_2)) \\
&= T(f^{-1}(CM_{C\times D})(X), f^{-1}(CM_{C\times D})(Y)).
\end{aligned}
$$

Thus

$$f^{-1}(CM_{C\times D})(XY) \geq T(f^{-1}(CM_{C\times D})(X), f^{-1}(CM_{C\times D})(Y)).$$

(2) Let $X = (g, h) \in G \times H$, then

$$f^{-1}(CM_{C\times D})(X^{-1}) = f^{-1}(CM_{C\times D})((g_1, h_1)^{-1})$$

$$
\begin{aligned}
&= CM_{C\times D}(f(g,h)^{-1})\\
&= CM_{C\times D}(f(g^{-1},h^{-1}))\\
&= CM_{C\times D}(f^{-1}(g),f^{-1}(h))\\
&= T(CM_C(f^{-1}(g)),CM_D(f^{-1}(h)))\\
&\geq T(CM_C(f(g)),CM_D(f(h)))\\
&= CM_{C\times D}(f(g),f(h))\\
&= CM_{C\times D}(f(g,h))\\
&= f^{-1}(CM_{C\times D})(g,h)\\
&= f^{-1}(CM_{C\times D})(X)
\end{aligned}
$$

and then $f^{-1}(CM_{C\times D})(X^{-1}) \geq f^{-1}(CM_{C\times D})(X)$. Thus $f^{-1}(C\times D) \in TFMS(G\times H)$. □

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