Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Differential operators and Narayana numbers

Jie Xiong[1] and Qi Fang[2,*]

[1] School of Mathematics,Northeastern University, Shenyang 110004, P. R. China.; xiongjiepapers@163.com
[2] School of Mathematics, Northeastern University, Shenyang 110004, P. R. China.
* Correspondence: qifangpapers@stumail.neu.edu.cn

Received: 1 January 2020; Accepted: 10 March 2020; Published: 30 March 2020.

Abstract: In this paper, we establish a connection between differential operators and Narayana numbers of both kinds, as well as a kind of numbers related to central binomial coefficients studied by Sulanke (Electron. J. Combin. 7 (2000), R40).

Keywords: Narayana numbers, recurrence relations, differential operators.

MSC: 05A05, 26A33.

1. Introduction

It is well known that the central binomial coefficients have the following expressions;

$$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2,\ \binom{2n+1}{n}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{k}.$$

For $0\leq k\leq n$, the *Narayana numbers* of types A are defined as;

$$N(n,k)=\frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

Let $N_n(x)=\sum_{k=0}^{n-1}N(n,k)x^k$ be the *Narayana polynomials* of types A (see [1]). It is well known that $N_n(x)$ is the rank-generating function of the lattice of non-crossing partition lattice with cardinality $\frac{1}{n+1}\binom{2n}{n}$ (see [2]). Hence the Catalan numbers have the following expression;

$$\frac{1}{n+1}\binom{2n}{n}=\sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

The *Narayana numbers* of type B are given as;

$$M(n,k)=\binom{n}{k}^2.$$

Let $M_n(x)=\sum_{k=0}^{n}M(n,k)x^k$. Reiner [2] showed that $M_n(x)$ is the rank-generating function of a ranked self-dual lattice with the cardinality $\binom{2n}{n}$.

Let $P(n,k)=\binom{n}{k}\binom{n+1}{k}$, and $S=\mathbb{P}\times\mathbb{P}$. According to [3, Proposition 1], $P(n,k)$ is the number of paths in $A_1(n+1)$ having $k+1$ steps, where $A_1(n)$ is the set of all lattice paths running from $(0;-1)$ to $(n;n)$ that use the steps in S and that remain strictly above the line $y=-1$ except initially.

The numbers $N(n,k), M(n,k)$ and $P(n,k)$ have been extensively studied. The readers are referred to [4] for details. In [5], *Daboul et al.*, reveals that

$$\frac{d^n}{dx^n}(e^{1/x})=(-1)^n e^{1/x}\sum_{k=1}^{n}\binom{n}{k}\binom{n-1}{k-1}(n-k)!x^{-n-k},$$

where the $\binom{n}{k}\binom{n-1}{k-1}(n-k)!$ are the *Lah numbers*. Motivated by this result, in this paper we show that the numbers $M(n,k), N(n,k)$ and $P(n,k)$ can be generated by higher-order derivative of functions of e^x. As an application, we obtain new recurrence relations for these classical combinatorial numbers.

2. Differential operators and Narayana polynomials

Let $P_n(x) = \sum_{k=0}^{n} P(n,k)x^{n-k}$, $Q_n(x) = \sum_{k=0}^{n} P(n,k)x^k$, then $Q_n(x) = x^n P_n(1/x)$. The first few $N_n(x), M_n(x)$ and $P_n(x)$ are listed as follows;

$$\begin{aligned}
N_1(x) &= 1,\ N_2(x) = 1+x, N_3(x) = 1+3x+x^2,\ N_4(x) = 1+6x+6x^2+x^3,\\
M_1(x) &= 1+x,\ M_2(x) = 1+4x+x^2, M_3(x) = 1+9x+9x^2+x^3,\\
P_1(x) &= 2+x,\ P_2(x) = 3+6x+x^2, P_3(x) = 4+18x+12x^2+x^3.
\end{aligned}$$

We define $\overline{N}(n,k) = (n+1)!n!N(n,k)$ and $\overline{M}(n,k) = n!^2 M(n,k)$. By using the explicit formulas of $\overline{N}(n,k)$ and $\overline{M}(n,k)$, it is routine to verify the following lemma.

Lemma 1. *For $0 \le k \le n+1$, we have*

$$\begin{aligned}
\overline{N}(n+1,k) &= ((n+1)(n+2)+2nk+k^2+3k)\overline{N}(n,k) + (4n+2n^2-2(k^2-1))\overline{N}(n,k-1)\\
&\quad + (n(n-1)-(k-2)(2n-k+1))\overline{N}(n,k-2),\\
\overline{M}(n+1,k) &= ((n+1)^2+2(n+1)k+k^2)\overline{M}(n,k) + (1+4n+2n^2-2k(k-1))\overline{M}(n,k-1)\\
&\quad + (n^2-(2n+2-k)(k-2))\overline{M}(n,k-2),
\end{aligned}$$

with initial conditions $\overline{N}(0,0) = \overline{M}(0,0) = 1$ and $\overline{N}(0,k) = \overline{M}(0,k) = 0$ for $k \neq 0$.

In the following discussion, let $D = \frac{d}{dx}$.

Theorem 1. *For $n \ge 1$, we have*

$$(De^xD)^n\left(\frac{1}{1-e^x}\right) = \frac{n!(n+1)!e^{(n+1)x}N_n(e^x)}{(1-e^x)^{2n+1}}, \tag{1}$$

$$(e^xD^2)^n\left(\frac{1}{1-e^x}\right) = \frac{n!^2 e^{(n+1)x}M_n(e^x)}{(1-e^x)^{2n+1}}, \tag{2}$$

$$(D^2e^x)^n\left(\frac{1}{1-e^x}\right) = \frac{n!^2 e^{nx}M_n(e^x)}{(1-e^x)^{2n+1}}. \tag{3}$$

Proof. Note that

$$\begin{aligned}
(De^xD)\left(\frac{1}{1-e^x}\right) &= \frac{2e^{2x}}{(1-e^x)^3},\\
(De^xD)^2\left(\frac{1}{1-e^x}\right) &= \frac{12e^{3x}(1+e^x)}{(1-e^x)^5},\\
(De^xD)^3\left(\frac{1}{1-e^x}\right) &= \frac{144e^{4x}(1+3e^x+e^{2x})}{(1-e^x)^7}.
\end{aligned}$$

Hence the formula (1) holds for $n = 1,2,3$. Assume that the result holds for n, where $n \ge 3$. Let $\overline{N}_n(x) = \sum_{k=0}^{n-1} \overline{N}(n,k)x^k$. Note that

$$(De^xD)^{n+1}\left(\frac{1}{1-e^x}\right) = (De^xD)\left(\frac{e^{(n+1)x}\overline{N}_n(e^x)}{(1-e^x)^{2n+1}}\right).$$

It follows that
$\overline{N}_{n+1}(x) = ((n+1)(n+2)+(4n+2n^2)x+n(n-1)x^2)\overline{N}_n(x) + (4x-6x^2+2x^3+2nx(1-x^2))D(\overline{N}_n(x))$
$+ x^2(1-x)^2D^2(\overline{N}_n(x))$.

Equating the coefficients of x^k in both sides, we immediately get the recurrence relation of $\overline{N}(n,k)$ given in Lemma 1. Therefore, the result holds for $n+1$.

Similarly, note that

$$\begin{aligned}(e^xD^2)\left(\frac{1}{1-e^x}\right) &= \frac{e^{2x}(1+e^x)}{(1-e^x)^3},\\ (e^xD^2)^2\left(\frac{1}{1-e^x}\right) &= \frac{4e^{3x}(1+4e^x+e^{2x})}{(1-e^x)^5},\\ (e^xD^2)^3\left(\frac{1}{1-e^x}\right) &= \frac{36e^{4x}(1+9e^x+9e^{2x}+e^{3x})}{(1-e^x)^7}.\end{aligned}$$

Hence the formula (2) holds for $n=1,2,3$. Assume it holds for n, where $n\geq 3$. Let $\overline{M}_n(x) = \sum_{k=0}^{n}\overline{M}(n,k)x^k$. Note that

$$(e^xD^2)^{n+1}\left(\frac{1}{1-e^x}\right) = (e^xD^2)\left(\frac{e^{(n+1)x}\overline{M}_n(e^x)}{(1-e^x)^{2n+1}}\right).$$

It follows that
$\overline{M}_{n+1}(x) = (1+x+n^2(1+x)^2+n(2+4x))\overline{M}_n(x) + (3x-4x^2+x^3+2nx(1-x^2))D(\overline{M}_n(x)) + x^2(1-x)^2D^2(\overline{M}_n(x))$.

Equating the coefficients of x^k in both sides, we immediately get the recurrence relation of $\overline{M}(n,k)$ given in Lemma 1. Therefore, the result holds for $n+1$. Along the same lines, it is routine to derive (3). This completes the proof. □

Note that $P(n,n-k) = \binom{n}{n-k}\binom{n+1}{n-k}$, then $P(n+1,n+1-k) = \binom{n+1}{n+1-k}\binom{n+2}{n+1-k}$.

It is easy to verify the following lemma;

Lemma 2. *For $0\leq k\leq n+1$, we have $(n+1)(n+2)P(n+1,n+1-k) = [(n+2)^2+(2n+5)k+k(k-1)]P(n,n-k) + [2(n^2+3n+1)-6(k-1)-2(k-1)(k-2)]P(n,n-k+1) + [n^2-(2n-1)(k-2)+(k-2)(k-3)]P(n,n-k+2)$.*

Theorem 2. *For $n\geq 1$, we have*

$$(D^2e^x)^n\frac{e^x}{(1-e^x)^2} = \frac{n!(n+1)!e^{(n+1)x}P_n(e^x)}{(1-e^x)^{2n+2}}, \tag{4}$$

$$(De^xD)^n\frac{e^x}{(1-e^x)^2} = \frac{n!(n+1)!e^{(n+1)x}Q_n(e^x)}{(1-e^x)^{2n+2}}. \tag{5}$$

Proof. Note that

$$\begin{aligned}(D^2e^x)\frac{e^x}{(1-e^x)^2} &= \frac{2e^{2x}(2+e^x)}{(1-e^x)^4},\\ (D^2e^x)^2\frac{e^x}{(1-e^x)^2} &= \frac{12e^{3x}(3+6e^x+e^{2x})}{(1-e^x)^6}.\end{aligned}$$

Hence the result holds for $n=1,2$. Assume that the result holds for n. Then from (4), we get the recurrence relation
$(n+1)(n+2)P_{n+1}(x) = [n^2x^2+(2+n)^2+2x(1+3n+n^2)]P_n(x) + x(1-x)[(2n-1)x+2n+5]P_n'(x) + x^2(1-x)^2P_n''(x)$.

Equating the coefficients of x^k in both sides, we get the recurrence relation of the numbers $P(n,n-k)$, which is given in Lemma 2, as desired. Along the same lines, one can derive (5). This completes the proof. □

By a change of variable $y=e^x$, we end our paper by giving a corollary;

Corollary 1. *For $n\geq 1$, let $D_y = \frac{d}{dy}$, we have*

1. $(yD_y y^2 D_y)^n \left(\frac{1}{1-y}\right) = \frac{n!(n+1)!y^{n+1}N_n(y)}{(1-y)^{2n+1}}$,
2. $(y^2 D_y y D_y)^n \left(\frac{1}{1-y}\right) = \frac{n!^2 y^{(n+1)}M_n(y)}{(1-y)^{2n+1}}$,
3. $(yD_y y D_y y)^n \left(\frac{1}{1-y}\right) = \frac{n!^2 y^n M_n(y)}{(1-y)^{2n+1}}$,
4. $(yD_y y D_y y)^n \frac{y}{(1-y)^2} = \frac{n!(n+1)!y^{(n+1)}P_n(y)}{(1-y)^{2n+2}}$,
5. $(yD_y y^2 D_y)^n \frac{y}{(1-y)^2} = \frac{n!(n+1)!y^{(n+1)}Q_n(y)}{(1-y)^{2n+2}}$.

Proof. It's not hard to verify the equations hold when $n = 1, 2$

$$(yD_y y^2 D_y)\left(\frac{1}{1-y}\right) = \frac{2y^2}{(1-y)^3},$$

$$(yD_y y^2 D_y)^2\left(\frac{1}{1-y}\right) = \frac{12y^3(1+y)}{(1-y)^5}.$$

Assume the result holds for m, where $m \geq 3$. Setting $y = e^x$, we get

$$\begin{aligned}(yD_y y^2 D_y)(yD_y y^2 D_y)^m\left(\frac{1}{1-y}\right) &= (e^x D_y e^{2x} D_y)\frac{m!(m+1)!y^{n+1}N_m(y)}{(1-y)^{2m+1}}\\ &= (De^x D)\frac{m!(m+1)!e^{(m+1)x}N_m(e^x)}{(1-e^x)^{2m+1}}\\ &= \frac{(m+1)!(m+2)!e^{(m+2)x}N_{m+1}(e^x)}{(1-e^x)^{2m+3}}\\ &= \frac{(m+1)!(m+2)!y^{m+1}N_{m+1}(y)}{(1-y)^{2m+3}}.\end{aligned}$$

Along the same lines, we can get the other statements. This completes the proof. □

Acknowledgments: The authors appreciate the careful review, corrections and helpful suggestions to this paper made by the referee.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Bonin, J., Shapiro, L., & Simion, R. (1993). Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths. *Journal of Statistical Planning and Inference, 34*(1), 35-55.

[2] Reiner, V. (1997). Non-crossing partitions for classical reflection groups. *Discrete Mathematics, 177*(1-3), 195-222.

[3] Sulanke, R. A. (2000). Counting lattice paths by Narayana polynomials. *The Electronic Journal of Combinatorics, 7*, R40.

[4] Sloane, N. J. (2008). The on-line encyclopedia of integer sequences. Published electronically at http://oeis.org.

[5] Daboul, S., Mangaldan, J., Spivey, M. Z., & Taylor, P. J. (2013). The Lah Numbers and the n th Derivative of $e^{\frac{1}{x}}$. *Mathematics Magazine, 86*(1), 39-47.