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Isometric Composite Integral Operators

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ABSTRACT

In this paper composite integral operators are studied. The necessary and sufficient conditions for composite integral operator minus identity operator to be isometry and unitary are obtained. It is shown that the set of all composite integral operators is algebra. The condition for composite integral operator and Volterra operator to commute is explored.

Keywords: Isometry; unitary; convolution kernel; expectation operator; Radon-nikodym derivative.

1. INTRODUCTION

Let (X,S,μ) be a σ-finite measure space and let ϕ: $X\to X$ be a non-singular measurable transformation $(\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0)$. Then a composition transformation $C_\phi:L^p(\mu) \to L^p(\mu)$ is defined by the equation $C_\phi f = fo\phi$ for every $f \in L^p(\mu)$. In case C_ϕ is continuous, we call it a composition operator induced by ϕ.

For each $f \in L^p(\mu)$, $1\leq p<\infty$, there exists a unique $\phi^{-1}(S)$ measurable function $E(f)$ such that

$$\int gfd\mu = \int gE(f)\, d\mu$$

for every $\phi^{-1}(S)$ measurable function g for which the left integral exits. The function E(f) is called conditional expectation of f with respect to the sub σ- algebra $\phi^{-1}(S)$. For $f \in L^p(\mu)$, E(f)

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= goϕ for exactly one S-measurable function g. We shall write g = E(f)oϕ^{-1}, which is well-defined measurable function.

Let K: X ×X → C be a measurable function. Then a linear transformation T_k :$L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$(T_k f)(x) = \int K(x,y) f(y) d\mu(y) \qquad \text{for all } f \in L^p(\mu)$$

is known as integral operator. The composite integral operator T_{k_ϕ} is a bounded linear operator,

$$T_{k_\phi} : L^p(\mu) \rightarrow L^p(\mu) \text{ defined by}$$

$$\text{as} \qquad (T_{k_\phi} f)(x) = \int K(x,y) f(\phi(y)) d\mu(y) \qquad \ldots\ldots\ldots.(1)$$

The equation (1) can also be written as

$$(T_{k_\phi} f)(x) = \int E(K_x)o\phi^{-1}(y) f_o(y) d\mu(y),$$

where $K_x(y)$ = K(x,y) and $f_o = \dfrac{d\mu\phi^{-1}}{d\mu}$, the Randon-nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ.

The integral operators and composite integral operators in particular Volterra integral operator on $L^p(\mu)$ have received considerable attention in recent years. The theory of integral operators is the source of all modern functional analysis and operator theory. Bloom and Kerman [1] have done great deal of work on integral operators. Campbell [2] has done lot of work on weighted composition operators. Gupta and Komal [3-6] also studied composite integral operators. Mathematicians like Halmos and Sunder [7] and Lyubic [8] studied integral operators and composition integration operators. Parathasarthy [9] has studied extensively expectation operators. An intensive study of composition operators is made over the past several decades. To worth mention, few of them are Ridge [10], Singh [11,12],Singh and Komal [13], Singh and Kumar [14], Singh and Manhas [15]. Setpanov [16,17], Whitley [18] established the Lyubic's conjecture and generalized it to Volterra composition operators on $L^p[0,1]$.

In this paper the necessary and sufficient conditions for composite integral operator minus identity operator to be isometry and unitary are obtained. It is shown that the set of all composite integral operators is algebra. The condition for compositie integral operator and Volterra operator to commute is also explored.

2. ISOMETRIC AND UNITARY OPERATOR

By $B(L^2(\mu))$, we denote the Banach space of all bounded linear operators from $L^2(\mu)$ into itself under the norm defined as

$|| f || = (\int_X | f |^2 d\mu)^{1/2}$ and $L^2(\mu) = \{f / f: X \to C$ is measurable and $\int | f |^2 d\mu < \infty\}$**.** In this section conditions for composite integral operator minus identity operator to be isometric and unitary are explored. Examples of an isometric composite integral operator are given.

Theorem 2.1: Let $T_{k_\phi} \in B(L^2(\mu))$ and k be real valued function. Then T_{k_ϕ} - I is an isometry on $L^2(X)$

if and only if

$$\int_X k_\phi(z,x)\, k_\phi(z,y)dz = k_\phi(x,y) + k_\phi(y,x) \quad \text{for allmost all } (x,y) \in X\times X.$$

Proof: For $T_{k_\phi} \in B(L^2(\mu))$, we have

$$(T_{k_\phi} - I)^* \circ (T_{k_\phi} - I) = I$$

$$[(T_{k_\phi})^* - I] \circ [T_{k_\phi} - I] = I$$

$$(T_{k_\phi})^* \circ T_{k_\phi} - (T_{k_\phi})^* - T_{k_\phi} + I = I$$

This implies that

$$(T_{k_\phi})^* \circ T_{k_\phi} = (T_{k_\phi})^* + T_{k_\phi}$$

Now, for $f \in L^2(\mu)$, we have a

$$T_{k_\phi} (T_{k_\phi})^* \circ f(x) = \int_X k_\phi (z,x)\, T_{k_\phi} f(z)\, d\mu(z)$$

$$= \int_X \int_X k_\phi (z,x)\, k_\phi (z,y)\, f(y)d\mu(y)\, d\mu(z) \quad \textbf{......}(2)$$

(using $(k_\phi(x, z))^* = k_\phi (z,x)$)

Also,

$$((T_{k_\phi})^* + T_{k_\phi}) f(x) = \int_X [k_\phi(y,x) + k_\phi(x,y)]\, f(y)\, d\mu(y) \quad ......(3)$$

From (2) and (3), we have

$$\int_X k_\phi(z,x)\, k_\phi(z,y)dz = k_\phi(x,y) + k_\phi (y,x) \quad \text{for allmost all } (x,y) \in X\times X.$$

Theorem 2.2: Let $T_{k_\phi} \in B(L^2(\mu))$ and k be real valued function. Then the following are equivalent:

(i) T_{k_ϕ} - I is an unitary on $L^2(X)$.

(ii) $(T_{k_\phi})^*$ - I is an unitary on $L^2(X)$.

(iii) $\int_X k_\phi(z,x)\, k_\phi(z,y)dz = \int_X k_\phi(x,z)\, k_\phi(y,z)dz$

$$= k_\phi(x,y) + k_\phi(y,x) \quad \text{for allmost all } (x,y) \in X\times X.$$

Proof: To prove T_{k_ϕ} - I is an unitary, it is enough to show that T_{k_ϕ} - I is a surjective isometry. The proof follows from theorem 2.1.

Example 2.3: Define K: $[0,1] \times [0,1] \rightarrow R$

as $K(x, y) = n(m + 1)\, x^m y^{n-1}$,

and $\phi : [0, 1] \rightarrow [0, 1]$as $\phi(x) = x^n$.

Suppose $T_{K_\phi} \in B(L^1([0, 1]))$. Then $\|T_{K_\phi} f\| = \| f \|$.

Since, $\|T_{K_\phi} f\| = \int_0^1 |(T_{K_\phi} f)(x)|\, d\mu(x)$.

We obtain

$$\|T_{K_\phi} f\| = \int_0^1 |(T_{K_\phi} f)(x)|\, d\mu(x).$$

$$= \int_0^1 \left| \int K(x, y)\, f(\phi(y))\, d\mu(y) \right| d\mu(x)$$

$$= \int_0^1 \left| \int n(m+1)\, x^m y^{n-1} f(y^n)\, d\mu(y) \right| d\mu(x).$$

$$= \int_0^1 \left| (m+1)x^m \int n\, y^{n-1} f(y^n)\, d\mu(y) \right| d\mu(x)$$

$$= \int_0^1 \left| (m+1)x^m \int f(t)\, dt \right| d\mu(x).$$

$$= (m+1) \int_0^1 |x^m| \, d\mu(x) \,.\| f\|.$$

$$= \| f \|.$$

Hence, T_{k_ϕ} is an isometry on $L^1([0,1])$.

Example 2.4: Suppose K: $[0,1] \times [0,1] \to R$ is defined as $K(x, y) = 4xy$
And $\phi : [0, 1] \to [0, 1]$ is defined as $\phi(x) = x^2$.

Then $\| T_{K_\phi} f\| = \| f \|$. Since, we have

$$\| T_{K_\phi} f\| = \int_0^1 |(T_{K_\phi} f)(x)| \, d\mu(x).$$

$$= \int_0^1 | \int K(x, y) \, f(\phi(y)) \, d\mu(y)| \, d\mu(x)$$

$$= \int_0^1 | \int 4\, x\, y\, f(y^2) \, d\mu(y)| \, d\mu(x).$$

$$= 2\int_0^1 | x \int 2\, yf(y^2) \, d\mu(y)| \, d\mu(x)$$

$$= 2\int_0^1 | x \int f(t) \, dt \,| \, d\mu(x).$$

$$= 2 \int_0^1 |x| \, d\mu(x) \,.\| f\|.$$

$$= \| f \|.$$

Hence, T_{k_ϕ} is an isometry on $L^1([0,1])$.

3. ALGEBRA OF COMPOSITE INTEGRAL OPERATORS

In this section it is shown that collection of composite integral operators is algebra. Product of composite integral operator is an integral operator induced convolution kernel. The condition for composite integral operator and Volterra to commute is explored.

Theorem 3.1: Let S = $\{T_{k_\phi} : T_{k_\phi} \in B(L^2(\mu))\}$. Then S is an algebra of $B(L^2(\mu))$.

Proof: Let $T_{k_\phi}, T_{h_\phi} \in B(L^2(\mu))$.

Then, for $f \in L^2(\mu)$, we have

$$(T_{k_\phi} + T_{h_\phi})\, f(x) = T_{k_\phi} f(x) + T_{h_\phi} f(x)$$

$$= \int_X k(x,y)\, f(\phi(y))\, d\mu(y) + \int_X h(x,y)\, f(\phi(y))\, d\mu(y)$$

$$= \int_X [\, k(x,y) + h(x,y)\,]\, f(\phi(y))\, d\mu(y)$$

$$= T_{(k+h)_\phi} f(x).$$

Also, for any scalar a, we have

$$(a T_{k_\phi} f)(x) = a \int_X k(x,y)\, f(\phi(y))\, d\mu(y)$$

$$= \int_X a\, k(x,y)\, f(\phi(y))\, d\mu(y)$$

$$= T_{a k_\phi} f(x).$$

Hence, the result is proved.

Theorem 3.2: Product of two composite integral operators is a composite integral operator induced by convolution kernel.

Proof: For $T_{k_\phi}, T_{h_\phi} \in B(L^2(\mu))$ and $f \in L^2(\mu)$, we have

$$T_{k_\phi} T_{h_\phi} f(x) = \int_X k_\phi(x,y)\, (T_{h_\phi} f)(y) d\mu(y)$$

$$= \int_X \int_X k_\phi(x,y)\, h_\phi(y,z)\, f(z) d\mu(z)\, d\mu(y)$$

$$= \int_X K_\phi(x,z)\, f(z) d\mu(z),$$

where $K_\phi(x,z) = (k_\phi * h_\phi)(x,z) = \int_X k_\phi(x,y)\, h_\phi(y,z)\, d\mu(y)$ is a convolution of two kernels k_ϕ and h_ϕ.

Theorem 3.3: Let $T_{k_\phi} \in B(L^2(\mu))$. Then composite integral operator T_{k_ϕ} and volterra operator V commute if and only if

$$\int_{y=t}^{1} k(x,y)\, d\mu(y) = \int_{y=0}^{x} k(y,t)\, d\mu(y)$$

Proof: For arbitrary $f \in L^2[0,1]$, we have

$$T_{k_\phi} Vf(x) = \int_0^1 k(x,y)\, (Vf)o\phi(y)\, d\mu(y)$$

$$= \int_0^1 k(x,y)\, Vf(\phi(y))\, d\mu(y)$$

$$= \int_{y=0}^{1} k(x,y) \left[\int_{t=0}^{y} f(\phi(t))\, d\mu(t) \right] d\mu(y)$$

$$= \int_{t=0}^{1} \left[f(\phi(t)) \int_{y=t}^{1} [k(x,y)\, d\mu(y)] d\mu(t) \right] \quad \ldots\ldots\ldots(4)$$

and $V T_{k_\phi} f(x) = \int_0^x (T_{k_\phi} f)(y)\, dy$

$$= \int_{y=0}^{x} \left[\int_{t=0}^{1} k(y,t)\, f(\phi(t))\, d\mu(t) \right] d\mu(y)$$

$$= \int_{t=0}^{1} f(\phi(t)) \left[\int_{y=0}^{x} k(y,t)\, d\mu(y) \right] d\mu(t) \quad \ldots\ldots\ldots\ldots.(5)$$

From (4) and (5), we have

$$\int_{y=t}^{1} k(x,y)\, d\mu(y) = \int_{y=0}^{x} k(y,t)\, d\mu(y).$$

4. CONCLUSION

In this paper an important criterion for isometric and unitary composite integral operator has been obtained. The theorem has been well illustrated with the help of examples. The results derived in this paper are quite helpful in studying other operator theoretic properties of composite integral operators.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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