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Three Solutions for a Navier Boundary Value System Involving the $(p(x), q(x))$-Biharmonic Operator

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Research Article

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Abstract

The existence of at least three weak solutions is established for a class of quasilinear elliptic systems involving the $(p(x), q(x))$-biharmonic operators with Navier boundary value conditions. The technical approach is mainly based on a three critical points theorem due to Ricceri [12].

Keywords: $(p(x), q(x))$*-biharmonic, Sobolev space, three critical points theorem.*
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1 Introduction

In this paper, we consider the problem of the tpye

$$\begin{cases} \triangle^2_{p(x)}u + e_p(x)|u|^{p(x)-2}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), \quad x \in \Omega, \\ \triangle^2_{q(x)}v + e_q(x)|v|^{q(x)-2}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), \quad x \in \Omega, \\ u = \triangle u = v = \triangle v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is a bounded domain with boundary of class C^1. $\lambda, \mu \geq 0$ are real numbers. $p(x), q(x) \in C^0(\overline{\Omega})$ with $\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$, $\max\{2, \frac{N}{2}\} < q^- \leq q^+$, $\triangle^2_{p(x)}u := \triangle(|\triangle u|^{p(x)-2}\triangle u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, which is a natural generalization of the p-biharmonic operator(where $p > 1$ is a constant).

We suppose that $F : \Omega \times R \times R \to R$ is a function such that $F(\cdot, s, t)$ is measurable in Ω for all $(s, t) \in R \times R$ and $F(x, \cdot, \cdot)$ is C^1 in $R \times R$ for a.e.$x \in \Omega$, F_s denotes the partial derivative of F with respect to s. For $G(x, s, t)$ and $e_p(x), e_q(x)$, we assume that the following conditions hold:

(G) $G : \Omega \times R \times R \to R$ is a Carathéodory function, $G(x, \cdot, \cdot)$ is C^1 in $R \times R$ for a.e.$x \in \Omega$ and $\sup_{\{|s|\leq\theta, |t|\leq\vartheta\}}(|G_s(\cdot, s, t)| + |G_t(\cdot, s, t)|) \in L^1(\Omega)$ for all $\theta, \vartheta > 0$;

(E) $e_p(x), e_q(x) \in L^\infty(\Omega)$ and ess $\inf_\Omega e_p(x)$, ess $\inf_\Omega e_q(x) > 0$, we denote $\|e_p\|_1 = \int_\Omega e_p(x)dx$ and $\|e_q\|_1 = \int_\Omega e_q(x)dx$.

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In [10], the authors studied the following super-linear p-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \triangle_p^2 u = g(x,u), & x \in \Omega, \\ u = \triangle u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

By means of Morse theory, the authors proved the existence of a nontrivial solution to (1.2) having a linking structure around the origin under the conditions: $\Omega \subseteq \mathbf{R}^N$ is bounded with smooth boundary, $N \geq 2p+1$, $g : \Omega \times R \to R$ is a Carathéodory function such that for some $C > 0, |g(x,t)| \leq C(1+|t|^{q-1})$ for a.e. $x \in \Omega$ and all $t \in R, 1 \leq q \leq p^* = \frac{Np}{N-2p}$. Moreover, in the case of both resonance near zero and non-resonance at ∞, the existence of two nontrivial solutions was obtained.

In [8,9], the authors considered the following problem:

$$\begin{cases} \triangle_p^2 u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \triangle u = 0 & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \triangle_p^2 u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & x \in \Omega, \\ \triangle_q^2 v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & x \in \Omega, \\ u = \triangle u = v = \triangle v = 0 & x \in \partial\Omega, \end{cases} \tag{1.4}$$

By the three critical points theorem obtained by Ricceri [12], they established the existence of three weak solutions to problem (1.3) and (1.4).

For more results for fourth-order elliptic equations with variable exponent, see [1-2, 18-22] and the reference therein.

The main purpose of the present paper is to prove the existence of at least three solutions of problem (1.1). We study problem (1) by using the three critical points theorem by B.Ricceri [12] too. On the basis of [3], we state an equivalent formulation of the three critical points theorem in [12] as follows.

Theorem 1. *Let X be a reflexive real Banach space; $\Phi : X \to R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to R$ a C^1 functional with compact Gâteaux derivative. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ for all $\lambda > 0$; and there are $r \in R$ and $u_0, u_1 \in X$ such that:
(ii) $\Phi(u_0) < r < \Phi(u_1)$;
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exists a non-empty open set $\Lambda \subseteq [0,\infty)$ and a positive real number ρ with the following property: for each $\lambda \in \Lambda$ and every C^1 functional $J : X \to R$ with compact Gâteaux derivative, there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0 \tag{1.5}$$

has at least three solutions in X whose norms are less than ρ.

The paper is organized as follows. In section 2, we recall some facts that will be needed in the paper. In section 3, we establish our main result.

2 Preliminaries

For the reader's convenience, we recall some background facts concerning the Lebesgue-Sobolev spaces with variable exponent and introduce some notations used below. For more details, we refer the reader to [4,7,13-14].

Set

$$C_+(\Omega) = \{h: \ h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $p(x) \in C_+(\Omega)$, define the space

$$L^{p(x)}(\Omega) = \{u | \ u \text{ is a measurable real-valued funcion, } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\{\lambda > 0 | \int_\Omega | \frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}.$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, and we call it variable exponent Lebesgue space.

The $W^{m,p(x)}(\Omega)$ is defined by

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) | \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \le m\},$$

where α is the multi-index and $|\alpha|$ is the order, m is a positive integer. $W^{m,p(x)}(\Omega)$ is a special class of so-called generalized Orlicz-Sobolev spaces. From [5], we know that $W^{m,p(x)}(\Omega)$ can be equipped with the norm $\|u\|_{W^{m,p(x)}(\Omega)}$ as Banach spaces, where

$$\|u\|_{W^{m,p(x)}(\Omega)} = \sum_{|\alpha| \le m} |D^\alpha u|_{p(x)}.$$

From [4], we know that spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces.

When $e_p(x)$ satisfies (E), we define

$$L^{p(x)}_{e_p(x)}(\Omega) = \{u | \ u \text{ is a measurable real-valued funcion, } \int_\Omega e_p(x)|u(x)|^{p(x)} dx < \infty\},$$

with the norm

$$|u|_{(p(x),e_p(x))} = \inf\{\lambda > 0 | \int_\Omega e_p(x) | \frac{u(x)}{\lambda}|^{p(x)} dx \le 1\},$$

then $L^{p(x)}_{e_p(x)}(\Omega)$ is a Banach space. Now we denote $X = X_p \times X_q$ where $X_p = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and $X_q = W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)$, where $W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, so does $W_0^{1,q(x)}(\Omega)$. For any $u \in X_p$, define

$$\|u\|_{e_p} = \inf\{\lambda > 0 | \int_\Omega | \frac{\triangle u(x)}{\lambda}|^{p(x)} + e_p(x)| \frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}.$$

Then it is easy to see that X_p endowed with the above norm is also a separable, reflexive Banach space.

Remark 1. According to [17], $\|\cdot\|_{W^{2,p(x)}(\Omega)}$ is equivalent to $|\triangle \cdot|_{p(x)}$ in X_p. Consequently, the norms $\|\cdot\|_{W^{2,p(x)}(\Omega)}$,$|\triangle \cdot|_{p(x)}$ and $\|u\|_{e_p}$ are equivalent.

In the following, we will use $\|\cdot\|_{e_p}$ to instead of $\|\cdot\|_{W^{2,p(x)}(\Omega)}$ on X_p. Similarly, we use $\|\cdot\|_{e_q}$ to instead of $\|\cdot\|_{W^{2,q(x)}(\Omega)}$ on X_q.

Proposition 1.(see [4,13]) *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \le (\frac{1}{p^-} + \frac{1}{(p^0)^-}) |u|_{p(x)} |v|_{p^0(x)} \le 2|u|_{p(x)}|v|_{p^0(x)}.$$

Proposition 2.(see [4,13]) *If we denote $\rho(u) = \int_\Omega |u|^{p(x)} dx, \ \forall u \in L^{p(x)}(\Omega)$, then*

(i) $|u|_{p(x)} < 1(=1;>1) \Leftrightarrow \rho(u) < 1(=1;>1)$;
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$; $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$;
(iii) $|u|_{p(x)} \to 0(\infty) \Leftrightarrow \rho(u) \to 0(\infty)$.
From Proposition 2, the following inequalities hold:

$$\|u\|^{p^-}_{e_p} \leq \int_\Omega | \triangle u(x)|^{p(x)} + e_p(x)|u(x)|^{p(x)}dx \leq \|u\|^{p^+}_{e_p}, \text{ if } \|u\|_{e_p} \geq 1; \tag{2.1}$$

$$\|u\|^{p^+}_{e_p} \leq \int_\Omega | \triangle u(x)|^{p(x)} + e_p(x)|u(x)|^{p(x)}dx \leq \|u\|^{p^-}_{e_p}, \text{ if } \|u\|_{e_p} \leq 1. \tag{2.2}$$

Proposition 3. *If $\Omega \subset \mathbf{R}^N$ is a bounded domain, then the imbedding $X_p \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $\frac{N}{2} < p^-$.*

Proof. It is well know that $X_p \hookrightarrow W^{2,p^-}(\Omega) \cap W^{1,p^-}_0(\Omega)$ is a continuous embedding, and the embedding $W^{2,p^-}(\Omega) \cap W^{1,p^-}_0(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact when $\frac{N}{2} < p^-$ and Ω is bounded. So we obtain the embedding $X_p \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $\frac{N}{2} < p^-$.□

From now on, the space X will be endowed with the norm

$$\|z\| = \|u\|_{e_p} + \|v\|_{e_q}, \text{ for any } z = (u, v) \in X.$$

Then X is a separable and reflexive Banach space. Naturally, we denote X^* by the space $(X_p \times X_q)^*$, the dual space of X.

From Proposition 3, we know that when $p^-, q^- > \frac{N}{2}$, the embedding $X \hookrightarrow C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ is compact, and there exists a positive constant c such that

$$\|z\|_\infty = \|u\|_\infty + \|v\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| + \sup_{x\in\overline{\Omega}} |v(x)| \leq c\|z\|. \tag{2.3}$$

3 Main Result

We define $\Phi, \Psi, J : X \to R$ by

$$\begin{aligned}\Phi(z) \quad = \quad & \int_\Omega \frac{1}{p(x)}(|\triangle u(x)|^{p(x)} + e_p(x)|u(x)|^{p(x)})dx \\ & + \int_\Omega \frac{1}{q(x)}(|\triangle v(x)|^{q(x)} + e_q(x)|v(x)|^{q(x)})dx,\end{aligned} \tag{3.1}$$

$$\Psi(z) = -\int_\Omega F(x, u, v)dx, \tag{3.2}$$

$$J(z) = -\int_\Omega G(x, u, v)dx. \tag{3.3}$$

Then for any $(\zeta, \eta) \in X$,

$$\begin{aligned}(\Phi'(z), (\zeta, \eta)) = \quad & \int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\zeta + e_p(x)|u|^{p(x)-2}u\zeta dx \\ + \quad & \int_\Omega |\triangle v|^{q(x)-2}\triangle v\triangle\eta + e_q(x)|v|^{q(x)-2}v\eta dx \ \ \forall z \in X,\end{aligned}$$

$$(\Psi'(z), (\zeta, \eta)) \quad = \quad -\int_\Omega F_u(x, u, v)\zeta dx - \int_\Omega F_v(x, u, v)\eta dx, \ \ \forall z \in X.$$

$$(J'(z),(\zeta,\eta)) \quad = \quad -\int_\Omega G_u(x,u,v)\zeta dx - \int_\Omega G_v(x,u,v)\eta dx, \ \ \forall z \in X.$$

We say that $z=(u,v) \in X$ is a weak solution of problem (1.1) if for any $(\zeta,\eta) \in X$

$$(\Psi'(z),(\zeta,\eta)) + \lambda(\Psi'(z),(\zeta,\eta)) + \mu(J'(z),(\zeta,\eta)) = 0.$$

Thus, we deduce that $z \in X$ is a weak solution of (1.1) if z is a solution of (1.5). It follows that we can seek for weak solutions of (1.1) by applying Theorem 1.

We first give the following result.

Lemma 1. *If Φ is defined in (3.1), then $(\Phi')^{-1}: X^* \to X$ exists and it is continuous.*

Proof. First, we show that Φ' is uniformly monotone. In fact, for any $\zeta,\eta \in \mathbf{R}^N$, we have the following inequality (see[6]):

$$(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta)(\zeta-\eta) \ge \frac{1}{2^p}|\zeta-\eta|^p, \ p \ge 2.$$

Thus, we deduce that

$$\begin{aligned}(\Phi'(z_1)-\Phi'(z_2), z_1-z_2) \quad &\ge \quad \min\{\frac{1}{2^{p^+}},\frac{1}{2^{q^+}}\}(\min\{\|u_1-u_2\|_{e_p}^{p^+}, \|u_1-u_2\|_{e_p}^{p^-}\} \\ &\quad + \min\{\|v_1-v_2\|_{e_q}^{q^+}, \|v_1-v_2\|_{e_q}^{q^-}\}),\end{aligned}$$

for any $z_1=(u_1,v_1), z_2=(u_2,v_2) \in X$, i.e.,$\Phi'$ is uniformly monotone.

From (2.1),(2.2), we can see that for any $z \in X$, we have that

$$\frac{(\Phi'(z),z)}{\|z\|} \ge \frac{\min\{\|u\|_{e_p}^{p^+}, |u\|_{e_p}^{p^-}\} + \min\{\|v\|_{e_q}^{q^+}, |v\|_{e_q}^{q^-}\}}{\|u\|_{e_p} + \|v\|_{e_q}}.$$

For $p^-, q^- \ge 2$, that's meaning Φ' is coercive on X.

By a standard argument, we know that Φ' is hemicontinuous. Therefore, the conclusion follows immediately by applying Theorem 26.A[16].□

To obtain our main result, we assume the following conditions on $F(x,s,t)$:

(A1) There exist $d(x) \in L^1(\Omega)$ and $0<\varsigma<p^-,\ 0<\tau<q^-$ such that

$$F(x,s,t) \le d(x)(1+|s|^\varsigma + |t|^\tau)$$

for a.e.$x \in \Omega$ and $(s,t) \in R \times R$;

(A2) $F(x,0,0)=0$ for a.e.$x \in \Omega$;

(A3) There exist $s_1, t_1 \in R$ with $|s_1|, |t_1| \ge 1$ such that

$$\text{meas}(\Omega) \sup_{(x,|s|,|t|)\in\Omega\times[0,ck_p]\times[0,ck_q]} F(x,s,t) \le \frac{(\frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+})\int_\Omega F(x,s_1,t_1)dx}{\frac{\|e_p\|_1}{p^-}|s_1|^{p^+} + \frac{\|e_q\|_1}{q^-}|t_1|^{q^+}}, \tag{3.4}$$

where c is given in (2.3) and

$$k_p = \max\{(\|e_p\|_1 + \frac{p^+\|e_q\|_1}{q^+})^{\frac{1}{p^+}}, (\|e_p\|_1 + \frac{p^+\|e_q\|_1}{q^+})^{\frac{1}{p^-}}\},$$

$$k_q = \max\{(\frac{q^+\|e_p\|_1}{p^+} + \|e_q\|_1)^{\frac{1}{q^+}}, (\frac{q^+\|e_p\|_1}{p^+} + \|e_q\|_1)^{\frac{1}{q^-}}\}.$$

(A3)$'$ $F(x,s,t)>0$ for any $x \in \Omega$ and $|s|$ or $|t|$ large enough, and there exist $M,N>0$ such that

$$F(x,s,t) \le 0, \ x \in \Omega, |s| \le M, |t| \le N;$$

Then we have the following main theorem.

Theorem 2. *Assume (A1),(A2),(A3)(or (A3)′),(G) and (E) hold. Then there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive real number ρ with the following property: for each $\lambda \in \Lambda$, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, problem (1.1) has at least three weak solutions whose norms are less than ρ.*

Proof. By the definitions of Φ, Ψ, J, we know that Ψ' is compact, Φ is weakly lower semi-continuous and bounded on each bounded subset of X. From lemma 1 we can see that $(\Phi')^{-1}$ is well defined, from condition (G), J is well defined and continuously G$\hat{a}$teaux differentiable on X, with compact derivative. Then we can use Theorem 1 to obtain the result. Now we show that the hypotheses of Theorem 1 are fulfilled.

Thanks to (A1), for each $\lambda \geq 0$, one has that

$$\lim_{\|z\| \to \infty} \Phi(z) + \lambda \Psi(z) = +\infty,$$

and so the assumption (i) of Theorem 1 holds.

Now we consider in two cases:

Case (i): (A3) holds, i.e., there exist $1 \leq |s_1|, |t_1|$ such that (3.4) hold.

Now we set $z_0 = (0, 0),\ z_1 = (s_1, s_1)$ and denote $r = \frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+} > 0$, then it is easy to see that

$$\Phi(z_1) > r > 0 = \Phi(z_0).$$

Thus, (ii) of Theorem 1 is satisfied.

At last, by (A2) we know $\Psi(z_0) = 0$, then

$$\begin{aligned} & \frac{(\Phi(z_1) - r)\Psi(z_0) + (r - \Phi(z_0))\Psi(z_1)}{\Phi(z_1) - \Phi(z_0)} \\ = \quad & r\frac{\Psi(z_1)}{\Phi(z_1)} \leq -r\frac{\int_\Omega F(x, s_1, t_1)dx}{\frac{|s_1|^{p^+}}{p^-}\|e_p\|_1 + \frac{|t_1|^{q^+}}{q^-}\|e_q\|_1}. \end{aligned} \tag{3.5}$$

On the other way, when $\Phi(z) \leq r$, we have

$$\min\{\|u\|_{e_p}^{p^+},\ \|u\|_{e_p}^{p^-}\} \leq rp^+,\ \min\{\|v\|_{e_q}^{p^+},\ \|v\|_{e_q}^{p^-}\} \leq rq^+.$$

We deduce that

$$\|u\|_{e_p} \leq \max\{(rp^+)^{\frac{1}{p^+}},\ (rp^+)^{\frac{1}{p^-}}\}$$

and

$$\|v\|_{e_q} \leq \max\{(rq^+)^{\frac{1}{q^+}},\ (rq^+)^{\frac{1}{q^-}}\}.$$

For $r = \frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+}$, then we have

$$\|u\|_{e_p} \leq k_p,\ \|v\|_{e_q} \leq k_q.$$

By (8), we obtain

$$\|u\|_\infty \leq ck_p,\ \|v\|_\infty \leq ck_q.$$

Thus, from (3.2), we have

$$\begin{aligned} -\inf_{z \in \Phi^{-1}((-\infty, r])} \Psi(z) \quad &= \quad \sup_{z \in \Phi^{-1}((-\infty, r])} -\Psi(z) \\ &\leq \quad \int_\Omega \sup_{(|u|,|v|) \in [0, ck_p] \times [0, ck_q]} F(x, u, v)dx \\ &\leq \quad \text{meas}(\Omega) \sup_{(x,|u|,|v|) \in \Omega \times [0, ck_p] \times [0, ck_q]} F(x, u, v) \end{aligned} \tag{3.6}$$

From (3.4)-(3.6) and the definition of r, we can see (iii) of Theorem 1 is hold.

Case (ii): (A3)$'$ holds. Then there exist $|s_2|, |t_2| > 1$ such that $F(x, s_2, t_2) > 0$ for any $x \in \Omega$ and $|s_2|^{p^-}\|e_p\|_1 \geq 1, |t_2|^{q^-}\|e_q\|_1 \geq 1$. Set $a = \min\{c, M\}$, $b = \min\{c, N\}$ then we have

$$\int_\Omega \sup_{(|s|,|t|)\in[0,a]\times[0,b]} F(x, s, t)dx \leq 0 < \int_\Omega F(x, s_2, t_2)dx. \tag{3.7}$$

We denote $z_2 = (s_2, t_2)$ and $r = \min\{\frac{1}{p^+}(\frac{a}{c})^{p^+},\ \frac{1}{q^+}(\frac{b}{c})^{q^+}\}$. Then it is easy to see that

$$\Phi(z_2) > r > \Phi(z_0).$$

So, (ii) of Theorem 1 is satisfied.

When $\Phi(z) \leq r$, similar to the above arguments, we obtain that

$$\|u\|_\infty \leq a,\ \|v\|_\infty \leq b. \tag{3.8}$$

At last, we see that

$$\begin{aligned} & \frac{(\Phi(z_2) - r)\Psi(z_0) + (r - \Phi(z_0))\Psi(z_2)}{\Phi(z_2) - \Phi(z_0)} \\ = \quad & r\frac{\Psi(z_2)}{\Phi(z_2)} \leq -r\frac{\int_\Omega F(x, s_2, t_2)dx}{\frac{|s_2|^{p^+}}{p^-}\|e_p\|_1 + \frac{|t_2|^{q^+}}{q^-}\|e_q\|_1} < 0. \end{aligned} \tag{3.9}$$

From (3.2) and (3.7), we have

$$\begin{aligned} -\inf_{z\in\Phi^{-1}((-\infty,r])} \Psi(z) &= \sup_{z\in\Phi^{-1}((-\infty,r])} -\Psi(z) \\ &\leq \int_\Omega \sup_{(|u|,|v|)\in[0,a]\times[0,b]} F(x, u, v)dx \leq 0. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we can see (iii) of Theorem 1 is still hold.

Then all the hypotheses of Theorem 1 are fulfilled. By Theorem 1, we know that there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0, \sigma]$, problem (1.1) has at least three weak solutions whose norms are less than ρ.□

By using Theorem 2, we have the following result.

Corollary 1. *Let $f, g : \Omega \times R \to R$ be Carathéodory functions, $\sup_{|\zeta|\leq s} |g(\cdot, \zeta)| \in L^1(\Omega)$ for all $s > 0$, and define $F(x, t) := \int_0^t f(x, y)dy$ for any $(x, t) \in \Omega \times R$, $e(x) \in L^\infty(\Omega)$ and ess$\inf_\Omega e(x) > 0$. Assume the following conditions hold.*

(B1) There exist $d(x) \in L^1(\Omega)$ and $0 < \varsigma < p^-$ such that

$$F(x, t) \leq d(x)(1 + |t|^\varsigma)$$

for a.e.$x \in \Omega$ and $t \in R$;

(B2) There exists $t_3 \in R$ with $|t_3| \geq 1$ such that

$$\textit{meas}(\Omega) \sup_{(x,|s|)\in\Omega\times[0,ck]} F(x, s) \leq \frac{p^-}{p^+}\frac{\int_\Omega F(x, t_3)dx}{|t_3|^{p^+}}, \tag{3.11}$$

where $c = \sup_{u\in X_p\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_e} < +\infty$ and

$$k = \max\{(\|e\|_1)^{\frac{1}{p^+}}, (\|e\|_1)^{\frac{1}{p^-}}\};$$

or

(B2)$'$ $F(x, t) > 0$ for any $x \in \Omega$ and $|t|$ large enough, and there exist $M > 0$ such that

$$F(x, t) \leq 0,\ x \in \Omega, |t| \leq M.$$

Then there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0, \sigma]$, the problem

$$\begin{cases} \triangle^2_{p(x)} u + e(x)|u|^{p(x)-2}u = \lambda f(x, u) + \mu g(x, u), & x \in \Omega, \\ u = \triangle u = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.12}$$

has at least three weak solutions whose norms are less than ρ.

Remark 2. If we take $e(x) \equiv 1$ in Ω, $\mu \equiv 0$, and replace $p(x)$-biharmonic operator by $p(x)$-Laplace operator, Corollary 1 becomes a version of Theorem 1 in [15], if we still have $f(x, t) = |t|^{\gamma(x)-2}t - t$ with $\gamma(x) \in C^0(\overline{\Omega})$ satisfies $2 < \gamma^- \leq \gamma^+ < p^-$, the problem was studied in [11] with Neumann conditions. Hence our Corollary 1 unifies and generalizes the main results in [8,11,15] to $p(x)$-biharmonic with Navier boundary value and our Theorem 2 generalizes the main result of [8,9,11,15] to the system (1).

At last, we give two examples.

Example 1. Let $\Omega = B(0, 1)$ be the unit ball of $\mathbf{R}^N$ with $N \geq 3$, set $p(x) = \frac{N}{2} + e^{|x|}, q(x) = \frac{N}{2} + 1 + \ln(1 + x^2), e_p(x) = (1 + x^2) = e_q(x), G(x, u, v) = x^2(u^2 + v^2)$ and

$$F(x, u, v) = \begin{cases} e^{x^2}(e^u + uv - 1), & x \in \Omega, u \leq M, v \in R, \\ e^{x^2}(ue^M + uv + \frac{1}{2}u^2 - Mu - (M-1)e^M + \frac{1}{2}M^2), & x \in \Omega, u > M, v \in R, \end{cases} \tag{3.13}$$

where M is a positive constant, i.e., we consider the following problem

$$\begin{cases} \triangle^2_{p(x)} u + (1 + x^2)|u|^{p(x)-2}u = \lambda f(x, u, v) + \mu 2x^2 u, & x \in \Omega, \\ \triangle^2_{q(x)} v + (1 + x^2)|v|^{q(x)-2}v = \lambda u + \mu 2x^2 v, & x \in \Omega, \\ u = \triangle u = v = \triangle v = 0, & x \in \partial\Omega. \end{cases} \tag{3.14}$$

where

$$f(x, u, v) = F_u(x, u, v) = \begin{cases} e^{x^2}(e^u + v), & x \in \Omega, u \leq M, v \in R, \\ e^{x^2}(e^M + v + u - M), & x \in \Omega, u > M, v \in R. \end{cases} \tag{3.15}$$

We can see that $p^+ = \frac{N}{2} + e, p^- = \frac{N}{2} + 1$, $q^+ = \frac{N}{2} + 1 + \ln 2, q^- = \frac{N}{2} + 1, \|e\|_1 = \frac{4}{3}$, and it is easy to see that for any $t_1 > 1$, there exists $s_1 > 1$ such that

$$\frac{(e^{s_1} + s_1 t_1 - 1)(\frac{N}{2} + 1)}{s_1^{\frac{N}{2}+e} + t_1^{\frac{N}{2}+1+\ln 2}} \geq \frac{e(e^{ck_p} + c^2 k_p k_q - 1)}{\frac{1}{\frac{N}{2}+e} + \frac{1}{\frac{N}{2}+1+\ln 2}}, \tag{3.16}$$

where $k_p = (\frac{4}{3} + \frac{4(\frac{N}{2}+1+\ln 2)}{3(\frac{N}{2}+e)})^{\frac{1}{\frac{N}{2}+1}}$, $k_q = (\frac{4}{3} + \frac{4(\frac{N}{2}+e)}{3(\frac{N}{2}+1+\ln 2)})^{\frac{1}{\frac{N}{2}+1}}$ are positive constants and c is given by (2.3). Then when $M \geq s_1$, $F(x, u, v)$ defined in (3.13) satisfies (A1)-(A3) of Theorem 2, and $G(x, u, v), e(x)$ satisfy (G) and (E) respectively, by Theorem 2, there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0, \sigma]$, system (3.14) has at least three weak solutions whose norms are less than ρ.

Example 2. Assume Ω,$p(x), q(x), e_p(x), e_q(x), G(x, u, v)$ are the same as in example 1, and suppose $N \geq 4$. Let

$$F(x, u, v) = (1 + 2x^2)(u^4v^2 + v^4u^2 - 2u^2v^2), \quad x \in \Omega, u, v \in R. \tag{3.17}$$

Obviously, $F(x, u, v)$ satisfies (A1) and (A2). By simple computation, we can see that

$$F(x, u, v) > 0, \quad \text{when } |u| > \sqrt{2} \text{ or } |v| > \sqrt{2}$$

and

$$F(x, u, v) < 0, \quad \text{when } |u| < 1 \text{ and } |v| < 1,$$

i.e., (A3)$'$ holds for $F(x,u,v)$ defined in (3.17).

Thus, there exist an open interval $\Lambda \subseteq [0,\infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0,\sigma]$, the system

$$\begin{cases} \triangle^2_{p(x)}u + (1+x^2)|u|^{p(x)-2}u = \lambda(4u^3v^2 + 2v^4u - 4uv^2) + \mu 2x^2u, & x \in \Omega, \\ \triangle^2_{q(x)}v + (1+x^2)|v|^{q(x)-2}v = \lambda(4v^3u^2 + 2u^4v - 4vu^2) + \mu 2x^2v, & x \in \Omega, \\ u = \triangle u = v = \triangle v = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.18}$$

has at least three weak solutions whose norms are less than ρ.

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Competing Interests

The authors declare that no competing interests exist.

References

[1] A.Ayoujil,A.R.El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent,Nonlinear Anal. 2009;71:4916-4926.
[2] A.EL Amrouss, F.Moradi,M.Moussaoui, Existence of solutions for fourth-order PDEs with variable exponents, Electron.J.Differential Equations. 2009;153:1-13.
[3] G.Bonanno, A minimax inequality and its applications to ordinary differential equations, J.Math.Anal.Appl. 2002;270:210-219.
[4] X.L.Fan,D.Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J.Math.Anal.Appl. 2001;263:424-446.
[5] H.Hudzik, On generalized Orlicz-Sobolev space, Funct.Approx. 1977;4:37-51.
[6] S.Kichenassamy, L.V$\acute{e}$ron, Singular solutions of the p-Laplace equation, Math.Ann. 1986;275:599-615.
[7] O.Ková$\check{c}$ik,J.Rákosník, On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$,Czechoslovak Math. J. 1991;41:592-618.
[8] C.Li,C.L.Tang, Three solutions for a Navier boundary value problem involving the p-biharmonic, Nonlinear Anal. 2010;72:1339-1347.
[9] L.Li,C.L.Tang, Existence of three solutions for (p,q)-biharmonic systems, Nonlinear Anal. 2010;73:796-805.
[10] S.Liu,M.Squassina, On the existence of solutions to a fourth-order quasilinear resonant problem, Abstr.Appl.Anal. 2002;7(3):125-133.
[11] M.Mih$\breve{a}$ilescu, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator,Nonlinear Anal. 2007;67:1419-1425.
[12] B.Ricceri, A three critical points theorem revisited, Nonlinear Anal. 2009;70:3084-3089.
[13] M.Ruzicka, Electrorheological Fluids:Modeling and Mathematical Theory, Lecture Notes in Math., Vol.1784, Springer-Verlag, Berlin; 2000.
[14] S.G.Samko, Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbf{R}^N)$, Dokl. Ross. Akad. Nauk. 1999;369(4):451-454.
[15] X.Y.Shi, X.H.Ding, Existence and multiplicity of solutions for a general $p(x)$-Laplacian Neumann problem,Nonlinear Anal. 2009;70:3715-3720.

[16] E.Zeidler, Nonlinear Functional Analysis and its Applications, II/B:Nonlinear Monotone Operators, Springer, New York; 1990.
[17] A.Zang, Y.Fu, Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces, Nonlinear Anal. 2008;69:3629-3636.
[18] S. Heidarkhani, Non-trivial solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems with Navier boundary conditions, Annales Polonici Mathematici. 2012;105(1):65-76.
[19] S. Heidarkhani, Y. Tian and C.-L. Tang, Existence of three solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems with Navier boundary conditions, Annales Polonici Mathematici. 2012;104:261-277.
[20] J.R. Graef, S.Heidarkhani and L. Kong, Multiple solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems, Comm. on Pure and Applied Analysis. 2013;12(3):1393-1406.
[21] C.Li, C.L. Tang, Three solutions for a Navier boundary value problem involving the p-biharmonic, Nonlinear Anal. 2010:72:1339-1347.
[22] C.Li, C.L. Tang, Existence of three solutions for (p, q)-biharmonic systems, Nonlinear Anal. 2010;73:796-805.

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