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Momentum-Entire Wavelets with Discrete Rotational Symmetries in 2D

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Abstract

We introduce wavelet bases consistent with the eigenspaces of the action of rotation by the angle $2\pi/N$ in dimension $d = 2$. Our particular construction yields wavelets that are *momentrum-entire* (a property weaker than the compact support property). The orthogonality of wavelets in a given eigenspace is based on an inner product that depends on the eigenspace, while the eigenspaces themselves form a *super-orthogonal system* over a certain family of Hilbert spaces. (We describe this notion in the Introduction.) The existence of a gradient-orthonormal basis of momentum-entire wavelets is an issue that remains open.

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1 Introduction

In the context of this paper, we say that an integrable function on $\mathbb{R}^d$ is *momentum-entire* if the analytic continuation of its Fourier transform is an entire function on $\mathbb{C}^d$. An obvious example of a momentum-entire function is a Gaussian on $\mathbb{R}^d$. Another example is a compactly supported continuous function on $\mathbb{R}^d$. In that case, the Fourier transform has an additional property - the growth of the analytic continuation in the imaginary directions is exponentially bounded. On the other hand, a continuous function on $\mathbb{R}^d$ with exponential decay is not necessarily momentum-entire. Roughly speaking, only real-analyticity of the Fourier transform is needed to guarantee the exponential localization.

This paper has been indirectly inspired by our quest [1] for gradient orthonormal bases of *compactly supported* wavelets in dimension $d > 1$. Such bases of *exponentially localized* wavelets were constructed 25 years ago [2] - at about the same time that the Lemarié bases [3], Meyer basis [4], and Daubechies bases [5] were constructed. The Lemarié wavelets are exponentially localized, the Meyer wavelet is a Schwartz function, and the Daubechies wavelets are compactly supported, but all of these bases are L^2-orthonormal instead of gradient-orthonormal. One may wonder about the more

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modest goal of constructing gradient-orthonormal bases of momentum-entire wavelets in dimension $d > 1$. Even this question is currently open, however.

Remark 1.1. Since Daubechies wavelets are compactly supported, they are momentum-entire wavelets. By contrast, Lemarié wavelets are not momentum-entire, as their Fourier transforms are only real-analytic. Meyer wavelets are not momentum-entire either. Indeed, their Fourier transforms are compactly supported, so one could say that Meyer wavelets are "position-entire" instead.

Here we are exclusively concerned with dimension $d = 2$, and discrete rotational symmetry will be a property of the new wavelets. Our wavelet construction will involve a machine for which Daubechies wavelets are the input and momentum-entire wavelets that lie in the eigenspaces for the action of a rotation are the output. For the sake of orientation, recall that in *two* dimensions each Daubechies basis is generated by *three* mother wavelets.

Our starting point is to consider a linear transformation $g \to Ug$ induced by a nonlinear mapping in momentum space - specifically,

$$\widehat{Ug} = \sqrt{2}\hat{g} \circ \Lambda, \tag{1.1}$$

$$\Lambda(\vec{k}) = (k_1^2 - k_2^2, 2k_1k_2). \tag{1.2}$$

Let H_1 denote the Hilbert space for gradient-orthogonality of functions over $\mathbb{R}^2$. More precisely, H_1 is the linear space of tempered distributions over $\mathbb{R}^2$ defined by the condition that $f \in H_1$ if and only if

$$\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^2 |\hat{f}(\vec{k})|^2 < \infty. \tag{1.3}$$

The inner product is given by

$$\begin{aligned} \langle f_1, f_2 \rangle_{H_1} &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^2 \hat{f}_1(\vec{k}) \overline{\hat{f}_2(\vec{k})} \\ &= \int\int_{\mathbb{R}^2} (\bigtriangledown f_1) \cdot (\bigtriangledown \bar{f}_2). \end{aligned} \tag{1.4}$$

The point is that Λ is a continuous mapping of $\mathbb{R}^2$ onto itself and that the Jacobian of Λ is given by

$$J_\Lambda(\vec{k}) = det \begin{bmatrix} 2k_1 & -2k_2 \\ 2k_2 & 2k_1 \end{bmatrix} = 4|\vec{k}|^2. \tag{1.5}$$

Clearly,

$$\begin{aligned} ||Ug||_{H_1}^2 &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^2 |\widehat{Ug}(\vec{k})|^2 \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\hat{g}(\Lambda(\vec{k}))|^2 J_\Lambda(\vec{k}) \\ &= \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 |\hat{g}(\vec{q})|^2 \\ &= ||g||_{L^2}^2, \end{aligned} \tag{1.6}$$

where the factor $\frac{1}{2}$ is cancelled by the double covering of $\mathbb{R}^2$ by Λ. Thus U is an isometry from $L^2(\mathbb{R}^2)$ into H_1.

Since the analytic continuation of Λ is an entire mapping from $\mathbb{C}^2$ onto itself, U preserves the momentum-entire property, but U is not unitary. It is obvious that the range of U consists of functions that are invariant with respect to reflection through the origin. In two dimensions this is equivalent to invariance with respect to 180° rotation. If we apply U to the nth-order Daubechies basis of $L^2(\mathbb{R}^2)$, we obtain an orthonormal set in H_1 consisting of momentum-entire wavelets, but it is not a basis of

H_1. Now the orthogonal complement of the functions with 180° rotational symmetry are those with 180° rotational anti-symmetry. It is easy to check that the differential operator

$$\bar{\partial} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}, \tag{1.7}$$

transforms the former type of function to the latter, and $\bar{\partial}$ obviously preserves the momentum-entire property. Unfortunately, $\bar{\partial}$ is also a unitary transformation of H_1 back to $L^2(\mathbb{R}^2)$, so our wavelet basis for the other eigenspace of the action of the 180° rotation is orthonormal in the initial Hilbert space $L^2(\mathbb{R}^2)$ instead of in the target space H_1. One way to stay in the target space is to apply the operator $|\bigtriangledown|^{-1}\bar{\partial}$ instead, but this operator destroys the momentum-entire property.

In what sense can one expand a function over a basis whose subsets have different orthogonality properties? For the case in point, the answer to this question lies in the observation that the two eigenspaces associated with the 180° rotation are mutually orthogonal with respect to both inner products. We generalize this scenario to a more abstract setting.

Let $\{H_\propto : \propto \in A\}$ be a family of Hilbert spaces that are also linear subspaces of the dual $\mathcal{T}'(\mathbb{R}^d)$ of some topological vector space $\mathcal{T}(\mathbb{R}^d)$ of test functions on $\mathbb{R}^d$. Consider a set $\{E_1, E_2, ..., E_N\}$ of linear operators on $\mathcal{T}'(\mathbb{R}^d)$ such that

$$E_l E_{l'} = \delta_{ll'} E_l, \qquad l, l' = 1, 2, ..., N, \tag{1.8}$$

$$\sum_{l=1}^{N} E_l f = f, \qquad f \in \mathcal{T}'(\mathbb{R}^d). \tag{1.9}$$

We say that $\{E_1, E_2, ..., E_N\}$ is a *super-orthogonal system* over $\{H_\propto : \propto \in A\}$ if the restriction of E_l to $H_\propto$ is an orthogonal projection of $H_\propto$ for every $\propto \in A$.

Now suppose for some set $\{\propto_1, \propto_2, ..., \propto_N\}$ of indices there is an $H_{\propto_l}$-orthonormal basis $\{h_{lj} : j \in J_l\}$ of the subspace $E_l(H_{\propto_l})$ of $H_{\propto_l}$ for each l. Then for

$$\varphi \in \bigcap_{l=1}^{N} H_{\propto_l}$$

we have

$$\begin{aligned} \varphi &= \sum_{l=1}^{N} E_l \varphi \\ &= \sum_{l=1}^{N} \sum_{j \in J_l} \langle E_l \varphi, h_{lj} \rangle_{H_{\propto_l}} h_{lj} \\ &= \sum_{l=1}^{N} \sum_{j \in J_l} \langle \varphi, h_{lj} \rangle_{H_{\propto_l}} h_{lj}, \end{aligned} \tag{1.10}$$

where the lth series converges in $H_{\propto_l}$. Such a composite expansion is unique as well. Indeed, if we have

$$\varphi = \sum_{l=1}^{N} \sum_{j \in J_l} c_{lj} h_{lj} \tag{1.11}$$

with the lth series converging in $H_{\propto_l}$, then

$$\begin{aligned} E_m \varphi &= \sum_{l=1}^{N} E_m \sum_{j \in J_l} c_{lj} h_{lj} \\ &= \sum_{j \in J_m} c_{mj} h_{mj} \end{aligned} \tag{1.12}$$

as a consequence of (1.8). Thus

$$\begin{aligned}\langle \varphi, h_{mj'} \rangle_{H_{\propto m}} &= \langle \varphi, E_m h_{mj'} \rangle_{H_{\propto m}} \\ &= \langle E_m \varphi, h_{mj'} \rangle_{H_{\propto m}} = c_{mj'} \end{aligned} \tag{1.13}$$

by the super-orthogonality property.

In this paper we construct wavelet bases of this type in dimension $d = 2$. Our topological vector space of test functions is the space $S_\infty(\mathbb{R}^2)$ of Schwartz functions on $\mathbb{R}^2$ whose moments vanish to all orders (so the dual space $S'_\infty(\mathbb{R}^2)$ properly contains the space of tempered distributions over $\mathbb{R}^2$). Our family of Hilbert spaces is a one-parameter continuum of massless Sobolev spaces, which we specify in the next section. The super-orthogonal system $\{E_1, E_2, ..., E_N\}$ will consist of the spectral projections of the action of rotation by the angle $2\pi/N$ (where E_N is the spectral projection for eigenvalue 1, and so we will set $E_N = E_0$). The point is that the action of rotation is unitary with respect to all of the massless Sobolev spaces, and therefore $\{E_0, E_1, ..., E_{N-1}\}$ is super-orthogonal with respect to this family of Hilbert spaces.

Our opening discussion has implicitly described our wavelet construction in the case $N = 2$, and we generalize it with a view to the requirement that the wavelets be momentum-entire. The differential operator $\bar{\partial}$ is ideal in this context. It obviously preserves the momentum-entire property, and it is a coherence operator in the sense that if we can build a wavelet set $\{h_{0j} : j \in J\}$ corresponding to E_0, then - as we will see later - the wavelet set $\{\bar{\partial}^l h_{0j} : j \in J\}$ corresponds to E_{N-l}. The catch is that $\bar{\partial}^l$ is not a unitary operator, but rather a unitary transformation from a given massless Sobolev space to another with a different degree.

The next section is devoted to generalizing the transformation U to the case where the output functions are invariant with respect to rotation by the angle $2\pi/N$. In Section 3 we describe the spectral resolution $\{E_0, E_1, ..., E_{N-1}\}$ arising from that rotation, and it is important to understand that this spectral analysis is just an exercise in the discrete Fourier transform that is independent of any function space inner product. In Section 4 we return to the $N = 2$ case and complete the description of the wavelet construction implied above. Section 5 covers the wavelet construction for arbitrary N, and we will see that the coherence with respect to scaling is based on the scale factor $\sqrt[N]{2}$ rather than 2. Moreoever, while scaling commutes with rotation, translation does not, so the familiar coherence with respect to integer-valued translations on the unit scale is replaced by a different coherence. Translation-invariant propagations generated by a couple of Nth-order differential operators take the place of those translations. (In the $N = 2$ case, they are directional Schrodinger propagations.)

2 The Transformations

For $-\infty < p < \infty$ we consider the Hilbert space H_p as a linear subspace of $S'_\infty(\mathbb{R}^2)$ defined by the condition that $f \in H_p$ if and only if

$$\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^{2p} |\hat{f}(\vec{k})|^2 < \infty. \tag{2.1}$$

In particular $H_0 = L^2(\mathbb{R}^2)$. An essential role will be played by the Cauchy-Riemann operator $\bar{\partial}$, which defines a unitary transformation from H_p to H_{p-1} for every p, but that is not our focus in this section.

Remark 2.1. For every $\epsilon > 0$, one can find an $f \in H_p$ such that

$$\hat{f}(\vec{k}) = O(|\vec{k}|^{\epsilon - 1 - p}), \qquad |\vec{k}| \to 0,$$

which means that for $p > 1$, one can find an element of H_p that is not a tempered distribution over $\mathbb{R}^2$. This is the reason why we chose a larger space of distributions. It is easy to verify that $H_p \subset S'_\infty(\mathbb{R}^2)$ for $-\infty < p < \infty$.

We are primarily interested in a family of isometries implemented by special nonlinear transformations in the momentum coordinates. For $N = 2, 3, 4, ...$ we define $\Lambda_N : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Lambda_2(\vec{k}) = (k_1^2 - k_2^2, 2k_1k_2), \tag{2.2}$$

$$\Lambda_3(\vec{k}) = (k_1^3 - 3k_1k_2^2, 3k_1^2k_2 - k_2^3), \tag{2.3}$$

$$\vdots$$

$$\Lambda_N(\vec{k}) = (\mathsf{Re}[(k_1 + ik_2)^N], \mathsf{Im}[(k_1 + ik_2)^N]), \tag{2.4}$$

$$\vdots$$

The Jacobian determinant J_N of Λ_N is easy to calculate:

$$\begin{aligned} J_N(\vec{k}) &= \det \begin{bmatrix} N\mathsf{Re}[(k_1+ik_2)^{N-1}] & N\mathsf{Re}[i(k_1+ik_2)^{N-1}] \\ N\mathsf{Im}[(k_1+ik_2)^{N-1}] & N\mathsf{Im}[i(k_1+ik_2)^{N-1}] \end{bmatrix} \\ &= \det \begin{bmatrix} N\mathsf{Re}[(k_1+ik_2)^{N-1}] & -N\mathsf{Im}[(k_1+ik_2)^{N-1}] \\ N\mathsf{Im}[(k_1+ik_2)^{N-1}] & N\mathsf{Re}[(k_1+ik_2)^{N-1}] \end{bmatrix} \\ &= N^2(\mathsf{Re}[(k_1+ik_2)^{N-1}])^2 + N^2(\mathsf{Im}[(k_1+ik_2)^{N-1}])^2 \\ &= N^2|(k_1+ik_2)^{N-1}|^2 \\ &= N^2|\vec{k}|^{2N-2}. \end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned} |\Lambda_N(\vec{k})|^2 &= (\mathsf{Re}[(k_1+ik_2)^N])^2 + (\mathsf{Im}[(k_1+ik_2)^N])^2 \\ &= |(k_1+ik_2)^N|^2 \\ &= |\vec{k}|^{2N}. \end{aligned} \tag{2.6}$$

Now introduce the mapping $g \mapsto U_N g$ defined by

$$\widehat{U_N g} = \sqrt{N}\hat{g} \circ \Lambda_N. \tag{2.7}$$

Clearly,

$$\begin{aligned} ||U_N g||_{H_q}^2 &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^{2q} |\widehat{U_N g}(\vec{k})|^2 \\ &= N \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}^{2q} |\hat{g}(\Lambda_N(\vec{k}))|^2 \\ &= \frac{1}{N} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^{2q-2N+2} |\hat{g}(\Lambda_N(\vec{k}))|^2 J_N(\vec{k}), \end{aligned} \tag{2.8}$$

so if we set

$$\Lambda_N(\vec{k}) = \vec{k}', \tag{2.9}$$

then

$$|\vec{k}| = |\vec{k}'|^{\frac{1}{N}}, \tag{2.10}$$

and therefore

$$||U_N g||_{H_q}^2 = \int_{-\infty}^{\infty} dk_2' \int_{-\infty}^{\infty} dk_1' |\vec{k}'|^{2\frac{q}{N} - 2 + \frac{2}{N}} |\hat{g}(\vec{k}')|^2, \tag{2.11}$$

where a factor of N appeared because the mapping Λ_N is an N-fold covering of $\mathbb{R}^2$. Thus U_N is an isometry from H_p into H_q, where

$$p = \frac{q}{N} - 1 + \frac{1}{N} \tag{2.12}$$

or rather

$$q = (p + 1)N - 1. \tag{2.13}$$

The only fixed exponent is $p = -1$, so H_{-1} is the natural Hilbert space in the sense that U_N is an operator on H_{-1} instead of a transformation from one space to another. However, we will be more interested in the fact that U_N is an isometry from $L^2(\mathbb{R}^2) = H_0$ into H_{N-1} as well.

These isometries are assuredly not unitary. Indeed, the ranges are the linear subspaces of interest to us because they can also be defined by discrete rotational symmetry. We pursue this in Section 3.

Remark 2.2. The way that we have defined the transformations Λ_N in the momentum coordinates may cause confusion (which may be further compounded by our intention to use the Cauchy-Riemann operator $\bar{\partial}$). After all, if we introduce the complex variable $z = k_1 + ik_2$, then Λ_N is effectively the transformation $z \mapsto z^N$ and the application of $\bar{\partial}$ is just multiplication by iz in momentum space. However, it is the analytic continuation of a Fourier transform in the separate variables k_1 and k_2 that we are concerned about. The fact that the mappings $z \mapsto z^N$ are entire does not directly concern us. We care about these special coordinate transformations on two counts:

(a) As nonlinear mappings, they are homogenous in the coordinates. This is important because wavelets involve scaling.

(b) The induced isometries U_N produce functions with discrete rotational symmetries.

The differential operator $\bar{\partial}$ also has a special effect on rotational symmetries, as we see in the next section.

Since partial differentiation preserves the momentum-entire property, the operator $\bar{\partial}$ clearly does so. Now for each $N = 2, 3, 4, ...$ the mapping $\Lambda_N : \mathbb{R}^2 \to \mathbb{R}^2$ is given by a two-component polynomial in two variables, so its analytic continuation in those two variables is obviously an entire mapping from $\mathbb{C}^2$ into itself. Therefore, U_N preserves the momentum-entire property for every N. It is important to note, however, that - in contrast to differential operators - these transformations do not preserve the compact support property.

3 Eigenspaces for a Rotation

For an arbitrary positive integer N, define the coordinate transformation $\Gamma_N : \mathbb{R}^2 \to \mathbb{R}^2$ as rotation by the angle $2\pi/N$ in the counter-clockwise direction. Thus - with $\vec{x} = (x_1, x_2)$ -

$$\Gamma_N(\vec{x}) = (x_1 \cos(2\pi/N) - x_2 \sin(2\pi/N), x_1 \sin(2\pi/N) + x_2 \cos(2\pi/N)). \tag{3.1}$$

Let W_N be the contravariant action of Γ_N on our distributions - i.e.,

$$W_N f = f \circ \Gamma_N, \tag{3.2}$$

where in the distributional sense,

$$(f \circ \Gamma_N)(\varphi) = f(\varphi \circ \Gamma_N^{-1}), \qquad \varphi \in S_\infty(\mathbb{R}^2). \tag{3.3}$$

It is obvious that W_N is a unitary operator on H_p for every p - since

$$\widehat{f \circ \Gamma_N} = \hat{f} \circ \Gamma_N^{-1} \tag{3.4}$$

and $|\vec{k}|^p$ is rotationally invariant - but the spectral analysis of W_N can be done in $S'_\infty(\mathbb{R}^2)$ with no inner product structure. The eigenvalues of W_N are the Nth roots of unity w_N^l, $l = 0, 1, ..., N-1$, where

$$w_N = \text{exp}(i2\pi/N). \tag{3.5}$$

If R_l^N denotes the lth eigenspace, then

$$R_l^N = \{f \in S'_\infty(\mathbb{R}^2) : f \circ \Gamma_N = w_N^l f\} \tag{3.6}$$

and the calculation of the projection onto R_l^N is an exercise in the discrete Fourier transform that is independent of the Hilbert space. Indeed, if we define

$$E_{N,l} f = \frac{1}{N} \sum_{m=0}^{N-1} \bar{w}_N^{lm} f \circ \Gamma_N^m, \qquad f \in S'_\infty(\mathbb{R}^2), \tag{3.7}$$

then $E_{N,l}$ is a super-orthogonal projection over the collection of Hilbert spaces H_p with $-\infty < p < \infty$.

To see that $E_{N,l}$ is a spectral projection of W_N, we first note that

$$\begin{aligned}
(E_{N,l} f) \circ \Gamma_N &= \frac{1}{N} \sum_{m=0}^{N-1} \bar{w}_N^{lm} f \circ \Gamma_N^{m+1} \\
&= \frac{1}{N} \sum_{m'=1}^{N} \bar{w}_N^{lm'-l} f \circ \Gamma_N^{m'} \\
&= \frac{\bar{w}_N^{-l}}{N} \left(\sum_{m'=1}^{N-1} \bar{w}^{lm'} f \circ \Gamma_N^{m'} + \bar{w}_N^{lN} f \circ \Gamma_N^N \right) \\
&= \frac{w_N^l}{N} \left(\sum_{m'=1}^{N-1} \bar{w}_N^{lm'} f \circ \Gamma_N^{m'} + f \right) \\
&= w_N^l E_{N,l} f,
\end{aligned} \tag{3.8}$$

so $E_{N,l} f \in R_l^N$ for every $f \in S'_\infty(\mathbb{R}^2)$. Second, we verify (1.8) by noting that for $f \in R_{l'}^N$,

$$\begin{aligned}
E_{N,l} f &= \frac{1}{N} \sum_{m=0}^{N-1} \bar{w}_N^{lm} f \circ \Gamma_N^m \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \bar{w}_N^{lm} w_N^{l'm} f = \delta_{ll'} f.
\end{aligned} \tag{3.9}$$

To establish that $E_{N,l}$ is a super-orthogonal projection, we show that the restriction of $E_{N,l}$ to H_p is self-adjoint in that Hilbert space. Notice that for $\varphi, \psi \in H_p$,

$$\begin{aligned}
\langle E_{N,l}\, \varphi, \psi \rangle_{H_p} &= \frac{1}{N} \sum_{m=0}^{N-1} \bar{w}_N^{lm} \langle \varphi \circ \Gamma_N^m, \psi \rangle_{H_p} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} w_N^{-lm} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |\vec{k}|^{2p} \hat{\varphi}(\Gamma_N^{-m}(\vec{k})) \overline{\hat{\psi}(\vec{k})} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} w_N^{-lm} \int_{-\infty}^{\infty} dk'_1 \int_{-\infty}^{\infty} dk'_2 |\vec{k}'|^{2p} \hat{\varphi}(\vec{k}') \overline{\hat{\psi}(\Gamma_N^m(\vec{k}'))} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} w_N^{-lm} \left\langle \varphi, \psi \circ \Gamma_N^{-m} \right\rangle_{H_p},
\end{aligned} \tag{3.10}$$

where we introduced the change of variable $\vec{k}' = \Gamma_N^{-m}(\vec{k})$ and used

$$|\vec{k}| = |\Gamma_N^m(\vec{k}')| = |\vec{k}'|. \tag{3.11}$$

Thus

$$\langle E_{N,l}\ \varphi, \psi\rangle_{H_p} = \left\langle \varphi, \frac{1}{N}\sum_{m=0}^{N-1} \bar{w}_N^{-lm}\psi \circ \Gamma_N^{-m} \right\rangle_{H_p}, \tag{3.12}$$

but on the other hand,

$$\begin{aligned}\frac{1}{N}\sum_{m=0}^{N-1} \bar{w}_N^{-lm}\psi \circ \Gamma_N^{-m} &= \frac{1}{N}\psi + \frac{1}{N}\sum_{m=1}^{N-1} \bar{w}_N^{lN-lm}\psi \circ \Gamma_N^{N-m} \\ &= \frac{1}{N}\psi + \frac{1}{N}\sum_{m'=1}^{N-1} \bar{w}^{lm'} N\psi \circ \Gamma_N^{m'} \\ &= E_{N,l}\psi. \end{aligned} \tag{3.13}$$

This completes the proof that the restriction of $E_{N,l}$ to H_p is a spectral projection of W_N in H_p for $-\infty < p < \infty$. Now the decomposition

$$H_p = \bigoplus_{l=0}^{N-l} \left(R_l^N \cap H_p\right) \tag{3.14}$$

is an automatic consequence of the spectral theory of self-adjoint operators, but it can also be verified independently. Indeed for $f \in S'_\infty(\mathbb{R}^2)$,

$$\begin{aligned}\sum_{l=0}^{N-1} E_{N,l} f &= \sum_{l=0}^{N-1}\frac{1}{N}\sum_{m=0}^{N-1} \bar{w}_N^{lm} f \circ \Gamma_N^m \\ &= \frac{1}{N}\sum_{m=0}^{N-1}\left(\sum_{l=0}^{N-1} \bar{w}_N^{lm}\right) f \circ \Gamma_N^m \\ &= \frac{1}{N}\sum_{m=0}^{N-1} N\delta_{m0} f \circ \Gamma_N^m = f \end{aligned} \tag{3.15}$$

so the condition (1.9) is satisfied.

It is very important to see the relation of the transformations defined in the previous section to these eigenspaces. For $\varphi \in H_p$ we have

$$\begin{aligned}\widehat{(U_N\varphi)\circ}\Gamma_N(\vec{k}) &= \widehat{U_N\varphi}(\Gamma_N^{-1}(\vec{k})) \\ &= \sqrt{N}\hat{\varphi}(\Lambda_N(\Gamma_N^{-1}(\vec{k})), \end{aligned} \tag{3.16}$$

but if we set $\Gamma_N^{-1}(\vec{k}) = \vec{k}'$,

$$\Lambda_N(\Gamma_N^{-1}(\vec{k})) = (\mathsf{Re}[(k_1' + ik_2')^N], \mathsf{Im}[(k_1' + ik_2')^N]), \tag{3.17}$$

while

$$k_1' = k_1\cos(2\pi/N) + k_2\sin(2\pi/N), \tag{3.18}$$

$$k_2' = -k_1\sin(2\pi/N) + k_2\cos(2\pi/N), \tag{3.19}$$

or rather

$$k_1' + ik_2' = w_N(k_1 + ik_2). \tag{3.20}$$

Therefore,

$$\begin{aligned}\Lambda_N(\Gamma_N^{-1}(\vec{k})) &= (\mathsf{Re}[w_N^N(k_1 + ik_2)^N], \mathsf{Im}[w_N^N(k_1 + ik_2)^N]) \\ &= (\mathsf{Re}[(k_1 + ik_2)^N], \mathsf{Im}[(k_1 + ik_2)^N]) \\ &= \Lambda_N(\vec{k}), \end{aligned} \tag{3.21}$$

from which it follows that

$$(U_N\varphi)\circ\Gamma_N = U_N\varphi. \tag{3.22}$$

We have already pointed out in the previous section that U_N is an isometry from H_p into H_q with

$$q = (p+1)N - 1. \tag{3.23}$$

We have now identified the isometric image as the eigenspace $R_0^N \cap H_q$ of the Γ_N-rotational operator W_N on H_q.

4 Construction in the $N = 2$ Case

In this case we are dealing with a $180°$ rotation only. The unitary transformations are given by

$$\widehat{U_2 g} = \sqrt{2}\hat{g}\circ\Lambda_2, \tag{4.1}$$

$$W_2 g = g\circ\Gamma_2, \tag{4.2}$$

where

$$\Lambda_2(\vec{k}) = (k_1^2 - k_2^2, 2k_1k_2), \tag{4.3}$$

$$\Gamma_2(\vec{x}) = -\vec{x}. \tag{4.4}$$

For $-\infty < p < \infty$, U_2 is an isometry from H_p into H_{2p+1} with isometric image

$$H_{2p+1}\cap R_0^2 = \{f\in H_{2p+1} : f\circ\Gamma_2 = f\}. \tag{4.5}$$

We now set $p = 0$. Actually $U_2 = U$, where U was discussed in the Introduction.

As we mentioned in the Introduction, our starting point is the nth-order Daubechies basis in two dimensions. Since it is orthonormal in $H_0 = L^2(\mathbb{R}^2)$, its image under U_2 is an orthonormal basis of $H_l \cap R_0^2$, but what are the new coherence properties? Since a Daubechies basis has three mother wavelets in two dimensions, we denote them by ψ_1, ψ_2, and ψ_3 so that

$$\mathcal{B} = \{2^r\psi_\mu(2^r\vec{x} - \vec{s}) : r\in\mathbb{Z}, \vec{s}\in\mathbb{Z}^2, \mu = 1,2,3\} \tag{4.6}$$

is the Daubechies basis. These functions are class $\mathbb{C}^{n-\epsilon}$ and all of their moments of order $\leq n$ vanish. The Fourier transform of this basis is given by

$$\hat{\mathcal{B}} = \{2^{-r}\text{exp}(-i2^{-r}\vec{s}\cdot\vec{k})\hat{\psi}_\mu(2^{-r}\vec{k}) : r\in\mathbb{Z}, \vec{s}\in\mathbb{Z}^2, \mu = 1,2,3\}, \tag{4.7}$$

so the Fourier transform of the image under U_2 is just

$$\widehat{U_2\mathcal{B}} = \{2^{-r}\sqrt{2}\text{exp}(-2^{-r}\vec{s}\cdot\Lambda_2(\vec{k}))\hat{\psi}_\mu(2^{-r}\Lambda_2(\vec{k})) : r\in\mathbb{Z}, \vec{s}\in\mathbb{Z}^2, \mu = 1,2,3\}. \tag{4.8}$$

On the other hand,

$$2^{-r}\Lambda_2(\vec{k}) = \Lambda_2(2^{-\frac{r}{2}}\vec{k}). \tag{4.9}$$

which implies

$$\sqrt{2}\hat{\psi}_\mu(2^{-r}\Lambda_2(\vec{k})) = \widehat{U_2\psi_\mu}(2^{-\frac{r}{2}}\vec{k}). \tag{4.10}$$

For notational convenience we set

$$U_2\psi_\mu(2^{\frac{r}{2}}\vec{x}) = \eta_{\mu,r}(\vec{x}), \tag{4.11}$$

and in two dimensions this means

$$2^{-r}\widehat{U_2\psi_\mu}(2^{-\frac{r}{2}}\vec{k}) = \widehat{\eta_{\mu,r}}(\vec{k}). \tag{4.12}$$

Thus

$$\widehat{U_2\mathcal{B}} = \{\text{exp}(-i2^{-r}\vec{s}\cdot\Lambda_2(\vec{k}))\widehat{\eta_{\mu,r}}(\vec{k}) : r\in\mathbb{Z}, \vec{s}\in\mathbb{Z}^2, \mu = 1,2,3\} \tag{4.13}$$

and therefore

$$U_2\mathcal{B} = \{\text{exp}(-i2^{-r}\vec{s}\cdot\Lambda_2(-i\bigtriangledown))\eta_{\mu,r} : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^2, \mu = 1,2,3\}. \tag{4.14}$$

This describes our orthonormal basis of $R_0^2 \cap H_1$.

There are two important points to made here. First, it is clear from (4.11) *that the scale factor is no longer* 2*, but rather* $\sqrt{2}$. Second, the propagator can be understood as a composition of directional Schrodinger propagators. Indeed,

$$\begin{aligned}\vec{s}\cdot\Lambda(i\bigtriangledown) &= s_1(-\partial_1^2 + \partial_2^2) - 2s_2\partial_1\partial_2\\ &= s_1(-\partial_1^2 + \partial_2^2) + s_2(\partial_-^2 - \partial_+^2),\end{aligned} \tag{4.15}$$

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_1 \pm \partial_2), \tag{4.16}$$

and so

$$\text{exp}(-i2^{-r}\vec{s}\cdot\Lambda_2(-i\bigtriangledown)) = \text{exp}(i2^{-r}s_1\partial_1^2)\text{exp}(-i2^{-r}s_1\partial_2^2)\text{exp}(-i2^{-r}s_2\partial_-^2)\text{exp}(i2^{-r}s_2\partial_+^2). \tag{4.17}$$

Indeed, an alternate way to describe our basis is to set

$$\text{exp}(is_1\partial_1^2)\text{exp}(-is_1\partial_2^2)\text{exp}(-is_2\partial_-^2)\text{exp}(is_2\partial_+^2)\eta_{\mu,0} = v_{\mu,\vec{s}} \tag{4.18}$$

at the unit scale and note that

$$U_2\mathcal{B} = \{v_{\mu,\vec{s}}(2^{\frac{r}{2}}\vec{x}) : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^2, \mu = 1,2,3\}. \tag{4.19}$$

The significance of this is apparent: translation-coherence on a given scale has been replaced by coherence with respect to a composition of certain translation-invariant propagations.

Obviously these expansion functions are momentum-entire because the Daubechies wavelets are momentum-entire. Equally obviously, this set is not complete, since a given function φ cannot be expanded in these functions unless $\varphi \circ \Gamma_2 = \varphi$. How do we find a set of expansion functions that covers the symmetry $\varphi \circ \Gamma_2 = -\varphi$ and still consists of momentum-entire expansion functions? The idea is that

$$(\bar{\partial}f) \circ \Gamma_2 = -\bar{\partial}(f \circ \Gamma_2), \qquad f \in S'_\infty(\mathbb{R}^2), \tag{4.20}$$

while $\bar{\partial}$ preserves the momentum-entire property. This relation means that

$$f \in \mathbb{R}_0^2 \Rightarrow (\bar{\partial}f) \circ \Gamma_2 = -\bar{\partial}f, \tag{4.21}$$

so $\bar{\partial}f \in \mathbb{R}_1^2$. On the other hand, $\bar{\partial}f$ does *not* lie in H_1. Since

$$\begin{aligned}||\bar{\partial}g||_{L^2}^2 &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 |(k_1 + ik_2)\hat{g}(\vec{k})|^2\\ &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 (k_1^2 + k_2^2)|\hat{g}(\vec{k})|^2 = ||g||_{H_1}^2,\end{aligned} \tag{4.22}$$

we know that $g \mapsto \bar{\partial}g$ is a unitary transformation from H_1 to H_0 and this implies it is also a unitary transformation from $\mathbb{R}_0^2 \cap H_1$ to $R_1^2 \cap H_0$.

Thus our total basis is $U_2\mathcal{B} \cup \bar{\partial}U_2\mathcal{B}$. For an arbitrary function $\varphi \in H_0 \cap H_1$ we write

$$\varphi = E_{2,0}\,\varphi + E_{2,1}\varphi, \tag{4.23}$$

where $\{E_{2,0}, E_{2,1}\}$ is the super-orthogonal system described in the previous section - for the $N = 2$ case. We exploit the memberships

$$E_{2,0}\varphi \in R_0^2 \cap H_1, \tag{4.24}$$

$$E_{2,1}\varphi \in R_1^2 \cap H_0 \tag{4.25}$$

as a consequence of the bases that are available. Accordingly, we apply the expansions

$$E_{2,0}\varphi = \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \langle E_{2,0}\varphi, v_{\mu,\vec{s},r} \rangle_{H_1} v_{\mu,\vec{s},r}, \tag{4.26}$$

$$E_{2,1}\varphi = \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \left\langle E_{2,1}\varphi, \bar{\partial} v_{\mu,\vec{s},r} \right\rangle_{L^2} \bar{\partial} v_{\mu,\vec{s},r}, \tag{4.27}$$

where

$$v_{\mu,\vec{s},r}(\vec{x}) = v_{\mu,\vec{s}}(2^{\frac{r}{2}}\vec{x}) \tag{4.28}$$

and therefore

$$\bar{\partial} v_{\mu,\vec{s},r}(\vec{x}) = 2^{\frac{r}{2}} \bar{\partial} v_{\mu,\vec{s}}(2^{\frac{r}{2}}\vec{x}). \tag{4.29}$$

Now by interspace orthogonality of the projections,

$$\langle E_{2,0}\varphi, v_{\mu,\vec{s},r} \rangle_{H_1} = \langle \varphi, v_{\mu,\vec{s},r} \rangle_{H_1}, \tag{4.30}$$

$$\left\langle E_{2,1}\varphi, \bar{\partial} v_{\mu,\vec{s},r} \right\rangle_{L^2} = \left\langle \varphi, \bar{\partial} v_{\mu,\vec{s},r} \right\rangle_{L^2}. \tag{4.31}$$

Combining this with (4.23), (4.26) and (4.27), we obtain the expansion

$$\varphi = \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \langle \varphi, v_{\mu,\vec{s},r} \rangle_{H_1} v_{\mu,\vec{s},r} + \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \left\langle \varphi, \bar{\partial} v_{\mu,\vec{s},r} \right\rangle_{L^2} \bar{\partial} v_{\mu,\vec{s},r} \tag{4.32}$$

with the understanding that the first sum converges in H_1 and the second sum converges in $H_0 = L^2(\mathbb{R}^2)$.

Examples of some equations are given below:

Definition 4.1. An Einstein manifold (E, g) intimated with a Divine structure (g, G) is called a Divine Einstein manifold. Thus a tripe (E, g, G) in which an Einstein metric g and a Divine structure G compatible with g are involved, is called a Divine Einstein manifold.

With the aid of the propositions (2.1) & (1.1) of [(**?**)] and [(**?**)] respectively, we can prove the following:

Proposition 4.1. *A Divine Einstein manifold* (E, g, G) *bears the property*

$$G^n = F_n G^i_j + F_{n-1}\delta^i_j, \tag{4.33}$$

for any integer number $n > 0$*. Here* $(F_n)_n$ *is the well known Fibonacci sequence.*

Proof. From equation (2.11), it is easy to get

$$G^3 = 2G^i_j + \delta^i_j,$$

and in general, if we suppose that

$$G^{n+1} = F_n G^2 + F_{n-1} G^i_j = (F_n + F_{n-1}) G^i_j + F_n \delta^i_j,$$

which due to Fibonacci properties evidently produces (4.33). □

Proposition 4.2. *The Divine Einstein structure* (g, G) *defined for an* n*-dimensional Einstein manifold* (E, g) *bears the trace property, given as*

$$\textit{trace}\,(G^2) = \textit{trace}\,(G) + n. \tag{4.34}$$

.

Proof. The equation (4.34) can be evidently derived from (2.11), if we operate (2.11) by trace operator and note that trace $\delta^i_j = n$. However, if we use the concept of orthonormal basis $(E_1, E_2, \cdots E_m)$ of tangent space $T_x(E)$ at a point $x \in E$ [(**?**)], we have from (2.12)

$$g(G^2 E_i, E_i) = g(GE_i, E_i) + g(E_i, E_i). \tag{4.35}$$

Summing (4.35) with respect to i, the equation (4.34) automatically set up. □

It is very clear that the proposition (2.5) of [(**?**)] also holds good in our case, i.e.,

5 Construction for Arbitrary N

In this general case, the angle of rotation is $2\pi/N$. The unitary transformations are given by

$$\widehat{U_N g} = \sqrt{N}\hat{g} \circ \Lambda_N, \tag{5.1}$$

$$W_N g = g \circ \Gamma_N, \tag{5.2}$$

where

$$\Lambda_N(\vec{k}) = (\text{Re}[(k_1 + ik_2)^N], \text{Im}[(k_1 + ik_2)^N]) \tag{5.3}$$

and Γ_N is given by (3.1). We are interested in U_N as an isometry from $H_0 = L^2(\mathbb{R}^2)$ to H_{N-1} with isometric image

$$H_{N-1} \cap R_0^N = \{f \in H_{N-1} : f \circ \Gamma_N = f\}, \tag{5.4}$$

and as before, we use the nth-order Daubechies basis in two dimensions as input.

This basis $\mathcal{B}$ is given by (4.6), and its image under U_N is an orthonormal basis of $H_{N-1} \cap R_0^N$. The Fourier transform of this image is given by

$$\widehat{U_N \mathcal{B}} = \{2^{-r}\sqrt{N}\text{exp}(i2^{-r}\vec{s} \cdot \Lambda_N(\vec{k}))\hat{\psi}_\mu(2^{-r}\Lambda_N(\vec{k})) : r \in \mathbb{X}, \vec{s} \in \mathbb{Z}^2, \mu = 1, 2, 3\}, \tag{5.5}$$

but since

$$2^{-r}\Lambda_N(\vec{k}) = \Lambda_N(2^{-\frac{r}{N}}\vec{k}), \tag{5.6}$$

we have

$$\sqrt{N}\hat{\psi}_\mu(2^{-r}\Lambda_N(\vec{k})) = \widehat{U_N\psi_\mu}(2^{-\frac{r}{N}}\vec{k}), \tag{5.7}$$

which means we may write

$$\widehat{U_N \mathcal{B}} = \{\text{exp}(-i2^{-r}\vec{s} \cdot \Lambda_N(\vec{k}))\widehat{\eta_{\mu,r}}(\vec{k}) : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^2, \mu = 1, 2, 3\}, \tag{5.8}$$

where we have set

$$2^{-r+\frac{2r}{N}} U_N \psi_\mu(2^{\frac{r}{N}}\vec{x}) = \eta_{\mu,r}(\vec{x}). \tag{5.9}$$

Thus

$$U_N \mathcal{B} = \{\text{exp}(-i2^{-r}\vec{s} \cdot \Lambda_N(-i\bigtriangledown))\eta_{\mu,r} : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^2, \mu = 1, 2, 3\} \tag{5.10}$$

is the orthonormal basis of $R_0^N \cap H_{N-1}$.

The scale factor is $\sqrt[N]{2}$ *rather than* 2. Indeed, it is straightforward to see that if we set

$$\text{exp}(-i\vec{s} \cdot \Lambda_N(-i\bigtriangledown))\eta_{\mu,0} = v_{\mu,\vec{s}}, \tag{5.11}$$

then

$$U_N \mathcal{B} = \{2^{-r+\frac{2r}{N}} v_{\mu,\vec{s}}(2^{\frac{r}{N}}\vec{x}) : r \in \mathbb{Z}, \vec{s} \in \mathbb{Z}^2, \mu = 1, 2, 3\}. \tag{5.12}$$

As for the propagation, the homogeneity of Λ_N implies

$$\Lambda_N(-i\bigtriangledown) = (-i)^N \Lambda_N(\bigtriangledown). \tag{5.13}$$

Since

$$\Lambda_N(\bigtriangledown) = (\mathsf{Re}[\bar{\partial}^N], \mathsf{Im}[\bar{\partial}^N]), \tag{5.14}$$

it follows that

$$v_{\mu,\vec{s}} = \mathsf{exp}((-i)^{N+1} s_1 \mathsf{Re}[\bar{\partial}^N])\mathsf{exp}((-i)^{N+1} s_2 \mathsf{Im}[\bar{\partial}^N])\eta_{\mu,0}. \tag{5.15}$$

In the $N = 2$ case we recover (4.18); in the $N = 3$ case we obtain

$$v_{\mu,\vec{s}} = \mathsf{exp}(s_1(\partial_1^3 - 3\partial_1\partial_2^2))\mathsf{exp}(s_2(3\partial_1^2\partial_2 - \partial_2^3))\eta_{\mu,0}. \tag{5.16}$$

In general, the order of the differential generators is N.

These expansion functions are momentum-entire, but they span only the subspace $R_0^N \cap H_{N-1}$ of H_{N-1} - i.e., only functions φ for which $\varphi \circ \Gamma_N = \varphi$ can be expanded in the functions $v_{\mu,\vec{s},r}$ where

$$v_{\mu,\vec{s},r}(\vec{x}) = 2^{-r+\frac{2r}{N}} v_{\mu,\vec{s}}(2^{\frac{r}{N}}\vec{x}). \tag{5.17}$$

The key to covering the other symmetries is to see that

$$\bar{\partial}(f \circ \Gamma_N) = w_N(\bar{\partial}f) \circ \Gamma_N, \qquad f \in S'_\infty(\mathbb{R}^2), \tag{5.18}$$

which means

$$(\bar{\partial}f) \circ \Gamma_N = \bar{w}_N\bar{\partial}(f \circ \Gamma_N). \tag{5.19}$$

We have already pointed out in previous sections that $\bar{\partial}$ preserves the momentum-entire property. The implication of (5.19) is that

$$\begin{aligned}(\bar{\partial}^m f) \circ \Gamma_N &= \bar{w}_N^m \bar{\partial}^m f \\ &= w_N^{N-m}\bar{\partial}^m f, \qquad f \in S'_\infty(\mathbb{R}^2).\end{aligned} \tag{5.20}$$

On the other hand, $\bar{\partial}^m$ is a unitary transformation from H_p to H_{p-m} for every real p - in particular, from H_{N-1} to H_{N-1-m}. The relation (5.20) supplies the additional information that $\bar{\partial}^m$ maps $R_0^N \cap H_{N-1}$ to $R_{N-m}^N \cap H_{N-1-m}$ and that $\bar{\partial}^m U_N \mathcal{B}$ is an orthonormal basis of the latter eigenspace of W_N in H_{N-1-m}.

Our total basis is

$$\bigcup_{m=0}^{N-1} \bar{\partial}^m U_N \mathcal{B}.$$

For an arbitrary function

$$\varphi \in \bigcap_{l=0}^{N-1} H_l$$

we write the resolution

$$\varphi = \sum_{m=0}^{N-1} E_{N,N-m}\varphi \tag{5.21}$$

where $\{E_{N,1}, E_{N,2}, ..., E_{N,N}\}$ is the super-orthogonal system described in Section 3. We exploit the memberships

$$E_{N,N-m}\varphi \in R_{N-m}^N \cap H_{N-m-1}, \qquad m = 0, 1, ..., N-1, \tag{5.22}$$

as a consequence of the bases we have available. Accordingly, we apply the expansions

$$E_{N,N-m}\varphi = \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \left\langle E_{N,N-m}\varphi, \bar{\partial}^m v_{\mu,\vec{s},r} \right\rangle_{H_{N-m-1}} \bar{\partial}^m v_{\mu,\vec{s},r}, \tag{5.23}$$

where the super-orthogonality of the projections yields

$$\left\langle E_{N,N-m}\varphi, \bar{\partial}^m v_{\mu,\vec{s},r} \right\rangle_{H_{N-m-1}} = \left\langle \varphi, \bar{\partial}^m v_{\mu,\vec{s},r} \right\rangle_{H_{N-m-1}}. \tag{5.24}$$

Combining this with (5.21) and (5.23), we obtain the expansion

$$\varphi = \sum_{m=0}^{N-1} \sum_{\mu=1}^{3} \sum_{\vec{s}\in\mathbb{Z}^2} \sum_{r\in\mathbb{Z}} \left\langle \varphi, \bar{\partial}^m v_{\mu,\vec{s},r} \right\rangle_{H_{N-m-1}} \bar{\partial}^m v_{\mu,\vec{s},r}, \tag{5.25}$$

where the mth series converges in H_{N-m-1}.

Remark 5.1. It is worth mentioning that one can obtain a basis of momentum-entire wavelets (with these symmetries) that is split between H_0 and H_1 only. Indeed, if N and m are even, we can use the unitary transformation

$$\varphi \to \triangle^{\frac{1}{2}(N-m-2)}\varphi$$

from H_{N-m-1} to H_1 to map the basis in $H_{N-m-1} \cap R^N_{N-m}$ to a basis in $H_1 \cap R^N_{N-m}$. The point is that the Laplacian $\triangle$ preserves both the momentum-entire property and the rotational symmetry property defined by R^N_{N-m}. Similarly, if N is even and m is odd, we use the unitary transformation

$$\varphi \to \triangle^{\frac{1}{2}(N-m-1)}\varphi$$

from H_{N-m-1} to $H_0 = L^2(\mathbb{R}^2)$ to map the basis in $H_{N-m-1} \cap R^N_{N-m}$ to a basis in $H_0 \cap R^N_{N-m}$. This transformation also applies to the case where N is odd and m is even, while the former transformation applies to the case where N and m are both odd.

6 Future Directions

This type of construction is worth pursuing in dimension $d > 2$, because the eigenspaces of discrete rotations have an interesting dependence on dimension. Indeed, we expect crystalline symmetries to play an important role. Another issue yet to be resolved is the existence of gradient-orthonormal bases of momentum-entire wavelets in multiple dimensions, including two dimensions.

Competing Interests

The author declares that no competing interests exist.

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