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Lie Algebra Structures Associated with Zero Curvature Equations and Generalized Zero Curvature Equations

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Research Article

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Abstract

Binary operations are introduced for triples satisfying zero curvature equations and quadruples satisfying generalized zero curvature equations, and it is shown that such operations define Lie algebra structures on the corresponding spaces of triples and quadruples.

Keywords: Zero curvature equation; Lie algebra; Symmetry

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1 Introduction

Zero curvature equations play a prominent role in constructing soliton equations (see, e.g., [1], [2]). By applying the trace identity [1] and the variational identity [3], Hamiltonian structures can be furnished for the resulting soliton equations, which generate infinitely many symmetries and conservation laws. The trace identity and the variational identity work for semisimple and non-semisimple matrix Lie algebras, respectively. Zero curvature equations associated with non-semisimple matrix Lie algebras yield integrable couplings [4]-[8], because a general complex Lie algebra has a semi-direct sum of a semisimple Lie algebra and a solvable Lie algebra (see, e.g., [9]). Integrable couplings are a pretty new research area in soliton theory [10], [11].

Let F be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $x, t \in \mathbb{R}$ be independent variables and $u = u(x,t)$ denote a q-dimensional column vector of F-valued dependent variables. Recall that the Gateaux derivative of an F-valued function $P[u] = P(x,t,u,u_x,\cdots)$ in a direction $v = v(x)$ is defined by

$$P'(u)[v] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P(x,t,u+\varepsilon v, u_x+\varepsilon v_x,\cdots), \tag{1.1}$$

where v is a q-dimensional column vector of F-valued functions. By $\mathcal{B}$, we denote all F-valued functions $P[u] = P(x,t,u,u_x,\cdots)$, which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x,t)$ as a vector function of x, and set

$$\mathcal{B}^r = \{(P_1, P_2, \cdots, P_r)^T \,|\, P_i \in \mathcal{B},\ 1 \le i \le r\},\ r \ge 1. \tag{1.2}$$

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By $\widetilde{\text{gl}}(r, F)$, we denote all $r \times r$ matrices $U = U(u, \lambda)$ in $\text{gl}(r, F)$ (the space of all $r \times r$ matrices with entries in F), which are C^∞-Gateaux differentiable with respect to $u = u(x, t)$ as a vector function of x and C^∞-differentiable with respect to λ.

For $K \in \mathcal{B}^q$, we consider an evolution equation

$$u_t = K[u] = K(x, t, u, u_x, \cdots). \tag{1.3}$$

If there are matrices $U = U(u, \lambda)$, $V = V(u, \lambda) \in \widetilde{\text{gl}}(r, F)$ and a C^∞-differentiable function f such that the following matrix equation [12] holds:

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0, \tag{1.4}$$

where the Gateaux derivative U' ($U' = (U'_{ij})_{r\times r}$ if $U = (U_{ij})_{r\times r}$) is assumed to be injective and U_λ and V_x are partial derivatives, then (U, V) is called a Lax pair of the equation (1.3) associated with the evolution law of the spectral parameter, $\lambda_t = f(\lambda)$.

The injective property of the Gateaux derivative U' guarantees that if $U'[K] = 0$, then $K = 0$. Therefore, if there is a Lax pair (U, V) associated with the evolution law of the spectral parameter, $\lambda_t = f(\lambda)$, then the equation (1.3) is equivalent to the compatibility condition

$$U_t - V_x + [U, V] = 0 \tag{1.5}$$

of the following spectral problems

$$\begin{cases} \varphi_x = U(u, \lambda)\varphi, \\ \varphi_t = V(u, \lambda)\varphi, \end{cases} \tag{1.6}$$

under the evolution law of the spectral parameter, $\lambda_t = f(\lambda)$ [12]. The equation (1.5) is called a zero curvature equation, and it presents a zero curvature representation of the evolution equation (1.3) while the matrix equation (1.4) with the injective Gateaux derivative U' holds (for the case of Lax representations, see [13]).

More generally, motivated by a Manakov pair [14], the evolution equation (1.3) may be represented as

$$U_t - V_x + [U, V] + W = 0, \tag{1.7}$$

or more concretely,

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] + W = 0 \tag{1.8}$$

for three matrices $U = U(u, \lambda)$, $V = V(u, \lambda)$, $W = W(u, \lambda) \in \widetilde{\text{gl}}(r, F)$ and a C^∞-differentiable function f, under the evolution law of the spectral parameter, $\lambda_t = f(\lambda)$. The equation (1.7) is called a generalized zero curvature equation, and it presents a generalized zero curvature representation of the evolution equation (1.3) while the matrix (1.8) with the injective Gateaux derivative U' holds.

Zero curvature representations are good tools for computing symmetries of soliton equations, and related algebraic structures help in recognizing diverse symmetry algebras [15] (for the case of Lax representations, see [16]). Generalized zero curvature representations will bring more possibilities to get symmetries and their algebras, since generalized zero curvature equations become zero curvature equations when $W = 0$. In this paper, we would like to show that there exist Lie algebra structures behind zero curvature equations and generalized zero curvature equations.

2 Lie algebra structures

Let $U = U(u, \lambda)$ be a given square matrix in the loop algebra $\widetilde{\text{gl}}(r, F)$. We do not need the injective property of the Gateaux derivative U' while addressing related Lie algebra structures in this section.

2.1 The case of zero curvature equations

Assume that $\mathcal{P}(U)$ denotes the set of all triples (V, K, f) satisfying (1.4), i.e.,

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U, V] = 0,$$

where $V = V(u, \lambda) \in \widetilde{\mathrm{gl}}(r, F)$, $K = K[u] \in \mathcal{B}^q$ and f is C^∞-differentiable with respect to λ. Obviously, $\mathcal{P}(U)$ forms a vector space on the field F under the usual addition and scalar multiplication:

$$(V_1, K_1, f_1) + (V_2, K_2, f_2) = (V_1 + V_2, K_1 + K_2, f_1 + f_2),\ \alpha(V, K, f) = (\alpha V, \alpha K, \alpha f), \qquad (2.1)$$

where (V_1, K_1, f_1), (V_2, K_2, f_2), $(V, K, f) \in \mathcal{P}(U)$ and $\alpha \in F$.

Theorem 2.1. *Let $U = U(u, \lambda)$ be a given square matrix in the loop algebra $\widetilde{gl}(r, F)$. The bracket $[\![-,-]\!]$ on the space $\mathcal{P}(U)$:*

$$[\![(V_1, K_1, f_1), (V_2, K_2, f_2)]\!] = ([\![V_1, V_2]\!], [K_1, K_2], [\![f_1, f_2]\!]), \qquad (2.2)$$

where

$$\begin{cases} [\![V_1, V_2]\!] = V_1'(u)[K_2] - V_2'(u)[K_1] + [V_1, V_2] + f_2V_{1,\lambda} - f_1V_{2,\lambda}, \\ [K_1, K_2] = K_1'(u)[K_2] - K_2'(u)[K_1], \\ [\![f_1, f_2]\!](\lambda) = f_{1,\lambda}(\lambda)f_2(\lambda) - f_1(\lambda)f_{2,\lambda}(\lambda) = f_1'(\lambda)f_2(\lambda) - f_1(\lambda)f_2'(\lambda), \end{cases} \qquad (2.3)$$

defines a Lie algebra structure on $\mathcal{P}(U)$.

Proof: It is direct to show that the bracket $[\![-,-]\!]$ is closed on the space $\mathcal{P}(U)$ (see also (12)). Obviously, it is bilinear and satisfies

$$[\![(V, K, f), (V, K, f)]\!] = 0,\ \forall\ (V, K, f) \in \mathcal{P}(U).$$

Therefore, one needs only prove the Jacobi identity [17]. To this end, we denote the first component of $[\![-,-]\!]$ by $[\![-,-]\!]_1$ and compute that

$$\begin{aligned} & [\![[\![(V_1, K_1, f_1), (V_2, K_2, f_2)]\!], (V_3, K_3, f_3)]\!]_1 \\ =\ & [\![([\![V_1, V_2]\!], [K_1, K_2], [\![f_1, f_2]\!]), (V_3, K_3, f_3)]\!]_1 \\ =\ & [\![V_1, V_2]\!]'[K_3] - V_3'[[K_1, K_2]] + [[\![V_1, V_2]\!], V_3] + f_3[\![V_1, V_2]\!]_\lambda - [\![f_1, f_2]\!]V_{3,\lambda} \\ =\ & (V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] + [V_1, V_2]'[K_3] + f_2(V_{1,\lambda})'[K_3] - f_1(V_{2,\lambda})'[K_3] \\ & -V_3'[[K_1, K_2]] + [V_1'(u)[K_2] - V_2'(u)[K_1] + [V_1, V_2] + f_2V_{1,\lambda} - f_1V_{2,\lambda}, V_3] \\ & +f_3(V_{1,\lambda}'[K_2] - V_{2,\lambda}'[K_1] + [V_1, V_2]_\lambda + f_{2,\lambda}V_{1,\lambda} + f_2V_{1,\lambda\lambda} - f_{1,\lambda}V_{2,\lambda} - f_1V_{2,\lambda\lambda}) \\ & -[\![f_1, f_2]\!]V_{3,\lambda}. \end{aligned}$$

We further group all the terms as follows:

$$\begin{aligned} & [\![[\![(V_1, K_1, f_1), (V_2, K_2, f_2)]\!], (V_3, K_3, f_3)]\!]_1 \\ =\ & \{(V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] - V_3'[[K_1, K_2]]\}_1 \\ & +\{[V_1, V_2]'[K_3] + [V_1'(u)[K_2] - V_2'(u)[K_1], V_3]\}_2 \\ & +\{f_2(V_{1,\lambda})'[K_3] - f_1(V_{2,\lambda})'[K_3] + f_3(V_{1,\lambda}'[K_2] - V_{2,\lambda}'[K_1])\}_3 \\ & +\{[[V_1, V_2], V_3]\}_4 + \{[f_2V_{1,\lambda} - f_1V_{2,\lambda}, V_3] + f_3[V_1, V_2]_\lambda\}_5 \\ & +\{f_3(f_{2,\lambda}V_{1,\lambda} - f_{1,\lambda}V_{2,\lambda}) - [\![f_1, f_2]\!]V_{3,\lambda}\}_6 + \{f_3(f_2V_{1,\lambda\lambda} - f_1V_{2,\lambda\lambda})\}_7 \\ \equiv\ & \sum_{i=1}^{7}\{T_i(1, 2, 3)\}_i. \end{aligned}$$

Noting (Ma, 1992b) that

$$(P'[K])'[S] - (P'[S])'[K] = P'[[K,S]],$$

we have

$$\begin{aligned} &T_1(1,2,3) + \text{cycle}(1,2,3) \\ &= (V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] - V_3'[[K_1,K_2]] \\ &\quad +(V_2'[K_3])'[K_1] - (V_3'[K_2])'[K_1] - V_1'[[K_2,K_3]] \\ &\quad +(V_3'[K_1])'[K_2] - (V_1'[K_3])'[K_2] - V_2'[[K_3,K_1]] \\ &= (V_1'[[K_2,K_3]] + V_2'[[K_3,K_1]] + V_3'[[K_1,K_2]]) \\ &\quad -(V_3'[[K_1,K_2]] + V_1'[[K_2,K_3]] + V_2'[[K_3,K_1]]) \\ &= 0. \end{aligned}$$

Since the commutator of matrices is a Lie bracket, we have

$$T_4(1,2,3) + \text{cycle}(1,2,3) = [[V_1,V_2],V_3] + [[V_2,V_3],V_1] + [[V_3,V_1],V_2] = 0.$$

It is also direct to check that

$$T_i(1,2,3) + \text{cycle}(1,2,3) = 0,\ i = 2,3,5,6,7.$$

This proves that the first component of the Jacobi identity holds. More easily, owe can show that the second and third components of the Jacobi identity hold (see also [12], [16]). Therefore, $[\![-,-]\!]$ satisfies the Jacobi identity, and so it defines a Lie algebra structure on the vector space $\mathcal{P}(U)$. □

2.2 The case of generalized zero curvature equations

Let $\mathcal{Q}(U)$ be the set of all quadruples (V,W,K,f) satisfying (1.8), i.e.,

$$U'(u)[K] + f(\lambda)U_\lambda - V_x + [U,V] + W = 0,$$

where $V = V(u,\lambda)$, $W = W(u,\lambda) \in \widetilde{\text{gl}}(r,F)$, $K = K[u] \in \mathcal{B}^q$ and f is C^∞-differentiable with respect to λ.

Obviously, $\mathcal{Q}(U)$ forms a vector space on the field F under the usual addition and scalar multiplication:

$$\begin{cases} (V_1,W_1,K_1,f_1) + (V_2,W_2,K_2,f_2) = (V_1+V_2, W_1+W_2, K_1+K_2, f_1+f_2), \\ \alpha(V,W,K,f) = (\alpha V, \alpha W, \alpha K, \alpha f), \end{cases} \tag{2.4}$$

where (V_1,W_1,K_1,f_1), (V_2,W_2,K_2,f_2), $(V,W,K,f) \in \mathcal{P}(U)$ and $\alpha \in F$. Let (V_1,W_1,K_1,f_1) and (V_2,W_2,K_2,f_2) be two elements in the space $\mathcal{Q}(U)$. Introduce the bracket

$$[\![(V_1,W_1,K_1,f_1),(V_2,W_2,K_2,f_2)]\!] = ([\![V_1,V_2]\!], [\![W_1,W_2]\!], [K_1,K_2], [\![f_1,f_2]\!]), \tag{2.5}$$

where

$$\begin{cases} [\![V_1,V_2]\!] = V_1'(u)[K_2] - V_2'(u)[K_1] + [V_1,V_2] + f_2 V_{1,\lambda} - f_1 V_{2,\lambda}, \\ [\![W_1,W_2]\!] = W_1'(u)[K_2] - W_2'(u)[K_1] + [W_1,V_2] - [W_2,V_1] + f_2 W_{1,\lambda} - f_1 W_{2,\lambda}, \\ [K_1,K_2] = K_1'(u)[K_2] - K_2'(u)[K_1], \\ [\![f_1,f_2]\!](\lambda) = f_{1,\lambda}(\lambda) f_2(\lambda) - f_1(\lambda) f_{2,\lambda}(\lambda) = f_1'(\lambda) f_2(\lambda) - f_1(\lambda) f_2'(\lambda), \end{cases} \tag{2.6}$$

motivated by an algebraic structure for Manakov's pairs in [14].

Theorem 2.2. *Let $U = U(u,\lambda)$ be a given square matrix in the loop algebra $\widetilde{gl}(r,F)$. The space $\mathcal{Q}(U)$ is closed under the bracket $[\![-,-]\!]$ determined by* (2.5) *and* (2.6).

Proof: Let (V_1, W_1, K_1, f_1) and (V_2, W_2, K_2, f_2) be two elements in the space $\mathcal{Q}(U)$, that is to say, we have

$$U'(u)[K_i] + f_i(\lambda)U_\lambda - V_{i,x} + [U, V_i] + W_i = 0,\ i = 1, 2. \tag{2.7}$$

First, it is direct to see that $[K_1, K_2] \in \mathcal{B}^q$, $[\![V_1, V_2]\!], [\![W_1, W_2]\!] \in \widetilde{gl}(r, F)$, and $[\![f_1, f_2]\!]$ is C^∞-differentiable. We would then like to prove that

$$U'(u)[[K_1, K_2]] + [\![f_1, f_2]\!](\lambda)U_\lambda - [\![V_1, V_2]\!]_x + [U, [\![V_1, V_2]\!]] + [\![W_1, W_2]\!] = 0. \tag{2.8}$$

It follows from (2.7) that

$$\begin{cases} (U'[K_1])'[K_2] + [U, V_1]'[K_2] = V'_{1,x}[K_2] - f_1U'_\lambda[K_2] - W'_1[K_2], \\ (U'[K_2])'[K_1] + [U, V_2]'[K_1] = V'_{2,x}[K_1] - f_2U'_\lambda[K_1] - W'_2[K_1], \end{cases} \tag{2.9}$$

and

$$\begin{cases} U'_\lambda[K_1] = V_{1,x\lambda} - [U, V_1]_\lambda - f_{1,\lambda}U_\lambda - f_1U_{\lambda\lambda} - W_{1,\lambda}, \\ U'_\lambda[K_2] = V_{2,x\lambda} - [U, V_2]_\lambda - f_{2,\lambda}U_\lambda - f_2U_{\lambda\lambda} - W_{2,\lambda}. \end{cases} \tag{2.10}$$

Let $\Theta = V'_1[K_2] - V'_2[K_1] + [V_1, V_2]$. Then we have

$$\Theta_x - [U, \Theta] = V'_{1,x}[K_2] - V'_{2,x}[K_1] + [V_{1,x}, V_2] + [V_1, V_{2,x}] - [U, V'_1[K_2] - V'_2[K_1] + [V_1, V_2]].$$

Note that we can compute that

$$\begin{aligned} &[U, [V_1, V_2]] = [V_1, [U, V_2]] - [V_2, [U, V_1]] \\ &= [V_1, V_{2,x} - U'[K_2] - f_2U_\lambda - W_2] - [V_2, V_{1,x} - U'[K_1] - f_1U_\lambda - W_1]. \end{aligned}$$

Therefore, we can further have

$$\begin{aligned} &\Theta_x - [U, \Theta] \\ &= V'_{1,x}[K_2] - V'_{2,x}[K_1] - [U, V'_1[K_2] - V'_2[K_1]] \\ &\quad + [V_1, U'[K_2] + f_2U_\lambda + W_2] - [V_2, U'[K_1] + f_1U_\lambda + W_1] \\ &= V'_{1,x}[K_2] - V'_{2,x}[K_1] - [U, V_1]'[K_2] + [U, V_2]'[K_1] \\ &\quad + f_2[V_1, U_\lambda] - f_1[V_2, U_\lambda] + [V_1, W_2] - [V_2, W_1] \\ &= U'[[K_1, K_2]] + f_1U'_\lambda[K_2] + W'_1[K_2] - f_2U'_\lambda[K_1] - W'_2[K_1] \\ &\quad + f_2[V_1, U_\lambda] - f_1[V_2, U_\lambda] + [V_1, W_2] - [V_2, W_1] \\ &= U'[[K_1, K_2]] + [\![f_1, f_2]\!]U_\lambda + f_1V_{2,x\lambda} - f_1[U, V_{2,\lambda}] \\ &\quad - f_2V_{1,x\lambda} + f_2[U, V_{1,\lambda}] + [\![W_1, W_2]\!], \end{aligned}$$

where the last two steps followed from (2.9) and (2.10), respectively. Now, this equality obviously implies that (2.8) holds. To conclude, the space $\mathcal{Q}(U)$ is closed under $[\![-,-]\!]$ defined by (2.5) and (2.6). □

It is also direct to see that the bracket $[\![(V_1, W_1, K_1, f_1), (V_2, W_2, K_2, f_2)]\!]$ is bilinear on the space $\mathcal{Q}(U)$. Therefore, the bracket $[\![(V_1, W_1, K_1, f_1), (V_2, W_2, K_2, f_2)]\!]$ defines an algebraic structure on the space $\mathcal{Q}(U)$. This, actually, defines a Lie algebra structure on the space $\mathcal{Q}(U)$.

Theorem 2.3. *Let $U = U(u,\lambda)$ be a given square matrix in the loop algebra $\widetilde{gl}(r,F)$. The bracket $[\![-,-]\!]$ determined by* (2.5) *and* (2.6) *defines a Lie algebra structure on the space $\mathcal{Q}(U)$.*

Proof: Obviously, the bilinear bracket $[\![-,-]\!]$ satisfies

$$[\![(V,W,K,f),(V,W,K,f)]\!]=0,\ \forall\ (V,W,K,f)\in\mathcal{Q}(U).$$

Therefore, one needs only prove the Jacobi identity [17]. Based on Theorem 2.1, it is sufficient to check the second component of the Jacobi identity. To this end, we denote the second component of $[\![-,-]\!]$ by $[\![-,-]\!]_2$ and we compute that

$$\begin{aligned}
&[\![[\![(V_1,W_1,K_1,f_1),(V_2,W_2,K_2,f_2)]\!],(V_3,W_3,K_3,f_3)]\!]_2\\
&=[\![([\![V_1,V_2]\!],[\![W_1,W_2]\!],[K_1,K_2],[\![f_1,f_2]\!]),(V_3,W_3,K_3,f_3)]\!]_2\\
&=[\![W_1,W_2]\!]'[K_3]-W_3'[[K_1,K_2]]+f_3[\![W_1,W_2]\!]_\lambda-[\![f_1,f_2]\!]W_{3,\lambda}\\
&\quad+[[\![W_1,W_2]\!],V_3]-[W_3,[\![V_1,V_2]\!]]\\
&=(W_1'[K_2])'[K_3]-(W_2'[K_1])'[K_3]+f_2W_{1,\lambda}'[K_3]-f_1W_{2,\lambda}'[K_3]\\
&\quad+[W_1'[K_3],V_2]+[W_1,V_2'[K_3]]-[W_2'[K_3],V_1]-[W_2,V_1'[K_3]]-W_3'[[K_1,K_2]]\\
&\quad+f_3(W_{1,\lambda}'[K_2]-W_{2,\lambda}'[K_1]+f_{2,\lambda}W_{1,\lambda}+f_2W_{1,\lambda\lambda}-f_{1,\lambda}W_{2,\lambda}-f_1W_{2,\lambda\lambda}\\
&\quad+[W_{1,\lambda},V_2]+[W_1,V_{2,\lambda}]-[W_{2,\lambda},V_1]-[W_2,V_{1,\lambda}])-[\![f_1,f_2]\!]W_{3,\lambda}\\
&\quad+[W_1'[K_2]-W_2'[K_1]+f_2W_{1,\lambda}-f_1W_{2,\lambda}+[W_1,V_2]-[W_2,V_1],V_3]\\
&\quad-[W_3,V_1'[K_2]-V_2'[K_1]+[V_1,V_2]+f_2V_{1,\lambda}-f_1V_{2,\lambda}].
\end{aligned}$$

We further group all the terms as follows:

$$\begin{aligned}
&[\![[\![(V_1,W_1,K_1,f_1),(V_2,W_2,K_2,f_2)]\!],(V_3,W_3,K_3,f_3)]\!]_2\\
&=\{(W_1'[K_2])'[K_3]-(W_2'[K_1])'[K_3]-W_3'[[K_1,K_2]]\}_1\\
&\quad+\{f_2W_{1,\lambda}'[K_3]-f_1W_{2,\lambda}'[K_3]+f_3(W_{1,\lambda}'[K_2]-W_{2,\lambda}'[K_1])\}_2\\
&\quad+\{[W_1'[K_3],V_2]-[W_2'[K_3],V_1]+[W_1'[K_2]-W_2'[K_1],V_3]\}_3\\
&\quad+\{[W_1,V_2'[K_3]]-[W_2,V_1'[K_3]]-[W_3,V_1'[K_2]-V_2'[K_1]]\}_4\\
&\quad+\{f_3(f_{2,\lambda}W_{1,\lambda}-f_{1,\lambda}W_{2,\lambda})-[\![f_1,f_2]\!]W_{3,\lambda}\}_5\\
&\quad+\{f_3(f_2W_{1,\lambda\lambda}-f_1W_{2,\lambda\lambda})\}_6\\
&\quad+\{f_3([W_{1,\lambda},V_2]-[W_{2,\lambda},V_1])+[f_2W_{1,\lambda}-f_1W_{2,\lambda},V_3]\}_7\\
&\quad+\{f_3([W_1,V_{2,\lambda}]-[W_2,V_{1,\lambda}])-[W_3,f_2V_{1,\lambda}-f_1V_{2,\lambda}]\}_8\\
&\quad+\{[[W_1,V_2]-[W_2,V_1],V_3]-[W_3,[V_1,V_2]]\}_9\\
&\equiv\sum_{i=1}^{9}\{S_i(1,2,3)\}_i.
\end{aligned}$$

We can directly prove that for all groups of terms, we have

$$S_i(1,2,3)+\text{cycle}(1,2,3)=0,\ 1\le i\le 9.$$

For example, we can readily check that

$$\begin{aligned}
&S_2(1,2,3)+\text{cycle}(1,2,3)\\
&=f_2W_{1,\lambda}'[K_3]-f_1W_{2,\lambda}'[K_3]+f_3(W_{1,\lambda}'[K_2]-W_{2,\lambda}'[K_1])\\
&\quad+f_3W_{2,\lambda}'[K_1]-f_2W_{3,\lambda}'[K_1]+f_1(W_{2,\lambda}'[K_3]-W_{3,\lambda}'[K_2])\\
&\quad+f_1W_{3,\lambda}'[K_2]-f_3W_{1,\lambda}'[K_2]+f_2(W_{3,\lambda}'[K_1]-W_{1,\lambda}'[K_3])\\
&=0,
\end{aligned}$$

and

$$\begin{aligned}
& S_5(1,2,3) + \text{cycle}(1,2,3) \\
&= f_3(f_{2,\lambda}W_{1,\lambda} - f_{1,\lambda}W_{2,\lambda}) - (f_{1,\lambda}f_2 - f_1 f_{2,\lambda})W_{3,\lambda} \\
&\quad + f_1(f_{3,\lambda}W_{2,\lambda} - f_{2,\lambda}W_{3,\lambda}) - (f_{2,\lambda}f_3 - f_2 f_{3,\lambda})W_{1,\lambda} \\
&\quad + f_2(f_{1,\lambda}W_{3,\lambda} - f_{3,\lambda}W_{1,\lambda}) - (f_{3,\lambda}f_1 - f_3 f_{1,\lambda})W_{2,\lambda} \\
&= 0.
\end{aligned}$$

Therefore, the bracket $[\![-,-]\!]$, defined by (2.5) and (2.6), satisfies the Jacobi identity, and so it defines a Lie algebra structure on the vector space $\mathcal{Q}(U)$. □

3 Concluding Remarks

Lie algebra structures were furnished on the vector space of triples satisfying zero curvature equations and the vector space of quadruples satisfying generalized zero curvature equations.

There are diverse symmetry algebras for soliton equations (see, e.g., [18]-[25]), because there exist Lie algebra structures behind their zero curvature representations. Generalized zero curvature representations should bring more possibilities for getting symmetries and their algebras for soliton equations.

Assume that the Gateaux derivative U' is injective. Then evidently, if two quadruples (V_1, W_1, K_1, f_1) and (V_2, W_2, K_2, f_2) satisfy

$$U'(u)[K_i] + f_i(\lambda)U_\lambda - V_{i,x} + [U, V_i] + W_i = 0,\ i = 1, 2,$$

then two autonomous evolution equations

$$u_t = K_1[u] = K_1(u, u_x, \cdots),\ u_t = K_2[u] = K_2(u, u_x, \cdots),$$

commute with each other, i.e.,

$$[K_1, K_2] = 0,$$

provided that

$$[\![V_1, V_2]\!] = 0,\ [\![W_1, W_2]\!] = 0,\ [\![f_1, f_2]\!] = 0.$$

Therefore, we can search for quadruples commuting with a given quadruple (V, W, K, f) to find symmetries for a given autonomous evolution equation

$$u_t = K[u] = K(u, u_x, \cdots).$$

Lie algebras are an important and beautiful area in mathematics [9], [17], and so, we expect that beautiful properties on symmetries of soliton equations can be explored, based on our Lie algebra structures established above.

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Competing Interests

The author declares that no competing interests exist.

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