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Estimation of Stress Strength Reliability $P[Y < X < Z]$ of Lomax Distribution under Different Sampling Scheme

Neethu Jacob [a*] and Anjana E. J. [b]

[a] *Department of Statistics, Nirmala College, Muvattupuzha, Kerala, India.*
[b] *Department of Mathematics, RheinischeFriedrich-Wilhelms-Universität Bonn, Bonn-53115, Germany.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

Lomax distribution can be considered as the mixture of exponential and gamma distribution. This distribution is an advantageous lifetime distribution in reliability analysis. The applicability of Lomax distribution is not restricted only to the reliability field, but it has broad applications in Economics, actuarial modelling, queuing problems, biological sciences, etc. Initially, Lomax distribution was proposed by Lomax in 1954, and it is also known as Pareto Type II distribution. Many statistical methods have been developed for this distribution; for a review of Lomax Distribution, see [1] and the references. The stress strength model plays an important role in reliability analysis. The term stress strength was first introduced by [2]. In the context of reliability, R is defined as the probability that the unit strength is greater than stress, that is, $R = P(X > Y)$, where X is the random strength of the unit, and Y is the instant stress applied to it. Thus, estimation of R is very important in Reliability Analysis.The estimates of R discussed in the context of Lomax distribution are limited to the study of a single stress strength model with upper stress. But in real life, there are situations where we have to consider not only the upper stress

*Corresponding author: E-mail: neethujacob3986@gmail.com;

but also the lower stress. Accordingly, in the present paper, the estimation of stress strength model $R = P(Y < X < Z)$ represents the situation where the strength X should be greater than stress Y and smaller than stress Z for Lomax distribution, Shrinkage maximum likelihood estimate and Quasi likelihood estimate are obtained both under complete and right censored data. We have considered the asymptotic confidence interval (CI) based on MLE and bootstrap CI for R. Monte Carlo simulation experiments were performed to compare the performance of estimates obtained.

Keywords: Lomax distribution; stress strength reliability; maximum likelihood estimator; quasi likelihood estimator; confidence interval.

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1 INTRODUCTION

The stress strength model plays an important role in reliability analysis. The term stress strength was first introduced by [2]. In the context of reliability, R is defined as the probability that the unit strength is greater than stress, that is $R = P[X > Y]$ where X is the random strength of the unit and Y is the instant stress applied to it. Moreover R provides the probability of system failure. The stress strength model was discussed in the literature from different point of view. Inference for generalized Lomax Distribution based on record statistics was considered by [3]. [4] studied the inference for the Lomax Distribution stress- strength model. [5] studied exponentiated Lomax Distribution.. The different stress strength model was considered by [6], [7], [8], [9]. Estimation of $R = P[X > Y]$ for Lomax Distribution with the presence of outliers was discussed by [10]. [11] studied the Power of Lomax Distribution with an application to bladder cancer data. The recent developments in stress strength reliability was discussed by [12], [13], [14], [15], [16], [17], [18], [19] and [20]. In this paper estimates the stress strength reliability for a component with a strength independent of opposite lower and upper bound stresses when the stresses and strength have Lomax distribution under different sampling schemes. Shrinkage maximum likelihood estimate and Quasi likelihood estimate are obtained under complete and right censored data. We have considered the asymptotic confidence interval (CI) based on MLE and bootstrap CI for R. Monte Carlo simulation experiments were performed to compare the performance of estimates obtained.

As a natural extension of the two component stress strength model we consider in the present paper the Maximum Likelihood Estimation (MLE) and Quasi likelihood estimate of stress strength reliability model $R = P[Y < X < Z]$, where X is the random strength and Y and Z are independent random stress variables follows Lomax Distribution. The stress strength model of $P[Y < X < Z]$ was studied in many branches of science such as Psychology, Medicine, Pedagogy, Engineering etc. The Estimation of $R = P[Y < X < Z]$ represents the situation where the strength X should be greater than stress Y and smaller than stress Z. For eg:- Many devices cannot function at high temperatures; neither can very low ones. Similarly a person's blood pressure should lie within two limits i.e systolic and diastolic. For instance many electronic components cannot work at high or low voltage. The estimate and the asymptotic confidence intervals are obtained for R under both complete and censored samples.

The Minimum Variance Unbiased (MVU), Maximum Likelihood and Empirical Estimator of $R = P[Y < X < Z]$ were discussed by [21]. [22] deal with the estimation of R when Y, Z and X are exponential random variables. Maximum Likelihood Estimate and Uniformly Minimum Variance Unbiased Estimate of R when Y, Z and X either uniform or exponential random variable with the unknown location parameter was considered by [23]. [24] focused on the estimate of $R = P[Y < X < Z]$, where Y and Z be a random stresses, and X be a random strength, having Weibull distribution in presence of k outliers. [25] focused on the estimate of $R = P[Y < X < Z]$,when Y, Z and X are independent and that these stress and strength variable follows Kumaraswamy Distribution. [26] discuss the estimation of Stress–Strength Reliability for $P[Y < X < Z]$ using Dagum Distribution. [27] the reliability of one strength- four stresses for Lomax Distribution was studied. Shrinkage estimation of stress strength reliability $P[Y < X < Z]$ for Lomax Distribution based on records was studied by [28].

A shrinkage estimator is a new estimate produced by shrinking a raw estimate. [29], [30] have given shrinkage estimates for population mean. [31] has found the shrinkage estimate of the parameters of exponential distribution. [32], [33], [34] obtain the Shrinkage estimation in the context of exponential distribution. [35] obtained the shrinkage estimator of stress strength reliability $R = P[X < Y]$ when X and Y are geometric distributions using record values.

The remaining part of this paper is organized in to seven sections. In section 3, we estimate the shrinkage estimate of R under complete sample scheme. In section 4, we estimate the shrinkage estimate of R based on right censored sample. Section 5 and 6 discuss the shrinkage estimate of the quasi likelihood function based on complete and censored sample. In Section 7 we illustrate estimator's performance by a simulation study, and finally, in Section 8, conclusions are made.

2 PRELIMINARY

Let X be the life of a device having an exponential distribution with a failure rate λ. It ia assumed that there could be some variation in λ value because of small fluctuations in the manufacturing tolerance (see [36]). This fluctuation is accommodated by assuming that λ have a gamma distribution with probability density function

$$f(\lambda|\alpha,\sigma) = \frac{\sigma^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\sigma\lambda}, \lambda \geq 0 \tag{2.1}$$

Then the density of X is obtained as

$$f(x) = \int_0^{\infty} \lambda e^{-\lambda x}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\sigma\lambda}d\lambda = \frac{\alpha}{\sigma}\left(1+\frac{x}{\sigma}\right)^{-(\alpha+1)}; x, \alpha, \sigma > 0 \tag{2.2}$$

which is Lomax Distribution. In other words, equipment is tested in the laboratory or ideal environment following exponential distribution when worked in the real environment, which is lighter or harsher than the laboratory environment, follows Lomax Distribution. So it is very important to consider the estimation problem of $P(Y < X < Z)$ when the underlying distribution follows Lomax Distribution.

Now let X be the strength of the random variable following Lomax distribution with parameters $L(\alpha_1, \lambda)$, where α_1 is the shape parameter and λ is scale parameter and Y and Z be the stress of the random variable following Lomax distribution with parameter $L(\alpha_2, \lambda)$ and $L(\alpha_3, \lambda)$ corresponding probability density functions are given below.

$$f(x, \alpha_1, \lambda) = \frac{\alpha_1\lambda^{\alpha_1}}{(x+\lambda)^{\alpha_1+1}}; x > 0, \alpha_1 > 0, \lambda > 0 \tag{2.3}$$

$$f(y, \alpha_2, \lambda) = \frac{\alpha_2\lambda^{\alpha_2}}{(y+\lambda)^{\alpha_2+1}}; y > 0, \alpha_2 > 0, \lambda > 0 \tag{2.4}$$

$$f(z, \alpha_3, \lambda) = \frac{\alpha_3\lambda^{\alpha_3}}{(y+\lambda)^{\alpha_3+1}}; z > 0, \alpha_3 > 0, \lambda > 0 \tag{2.5}$$

Under this situation the stress strength reliability

$$\begin{aligned} R = P[Y < X < Z] &= \int_0^{\infty} F_y(x)\overline{F}_z(x)f(x)\,dx = \int_0^{\infty} F_y(x)[1-F_z(x)]f(x)\,dx \\ &= \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}; 0 < R < 1 \end{aligned} \tag{2.6}$$

In the present paper we assumes that the scale parameter λ which is common for all the three variables is known.

3 MAXIMUM LIKELIHOOD ESTIMATION OF R BASED ON COMPLETE SAMPLE

Let $\underline{x} = (x_1, x_2, ..., x_{n_1})$ be the random sample of n_1 observation taken from Lomax distribution $L(\alpha_1, \lambda)$ then its likelihood function is given by

$$L(\underline{x}|\alpha_1, \lambda) = \prod_{i=1}^{n_1} \frac{\alpha_1 \lambda^{\alpha_1}}{(x_i + \lambda)^{\alpha_1+1}} = \alpha_1^{n_1} \lambda^{n_1\alpha_1} \prod_{i=1}^{n_1} (x_i + \lambda)^{-(\alpha_1+1)} \tag{3.1}$$

Let $\underline{y} = (y_1, y_2, ..., y_{n_2})$ be the random sample of n_2 observation taken from Lomax Distribution $L(\alpha_2, \lambda)$ then its likelihood function is given by

$$L(\underline{y}|\alpha_2, \lambda) = \prod_{j=1}^{n_2} \frac{\alpha_2 \lambda^{\alpha_2}}{(y_j + \lambda)^{\alpha_2+1}} = \alpha_2^{n_2} \lambda^{n_2\alpha_2} \prod_{j=1}^{n_2} (y_j + \lambda)^{-(\alpha_2+1)} \tag{3.2}$$

Let $\underline{z} = (z_1, z_2, ..., z_{n_3})$ be the random sample of n_3 observation taken from Lomax Distribution $L(\alpha_3, \lambda)$ then its likelihood function is given by

$$L(\underline{z}|\alpha_3, \lambda) = \prod_{k=1}^{n_3} \frac{\alpha_3 \lambda^{\alpha_3}}{(z_k + \lambda)^{\alpha_3+1}} = \alpha_3^{n_3} \lambda^{n_3\alpha_3} \prod_{k=1}^{n_3} (z_k + \lambda)^{-(\alpha_3+1)} \tag{3.3}$$

The joint likelihood function is given by

$$\begin{aligned} L(\underline{x}, \underline{y}, \underline{z}|\alpha_1, \alpha_2, \alpha_3, \lambda) =& \alpha_1^{n_1} \lambda^{n_1\alpha_1} \prod_{i=1}^{n_1} (x_i + \lambda)^{-(\alpha_1+1)} \alpha_2^{n_2} \lambda^{n_2\alpha_2} \prod_{j=1}^{n_2} (y_j + \lambda)^{-(\alpha_2+1)} \\ & \times \alpha_3^{n_3} \lambda^{n_3\alpha_3} \prod_{k=1}^{n_3} (z_k + \lambda)^{-(\alpha_3+1)} \end{aligned} \tag{3.4}$$

Taking Logarithm on both side of (3.4) we get

$$\begin{aligned} logL =& n_1 log\alpha_1 + n_1\alpha_1 log\lambda - (\alpha_1 + 1) \sum_{i=1}^{n_1} log(x_i + \lambda) + n_2 log\alpha_2 + n_2\alpha_2 log\lambda - (\alpha_2 + 1) \sum_{j=1}^{n_2} log(y_j + \lambda) \\ & + n_3 log\alpha_3 + n_3\alpha_3 log\lambda - (\alpha_3 + 1) \sum_{k=1}^{n_3} log(z_k + \lambda) \end{aligned} \tag{3.5}$$

Differentiating (3.5) partially with respect to α_1, α_2 and α_3 and equating to zero we get the MLE of α_1, α_2 and α_3 as

$$\hat{\alpha_1}_{mle} = \frac{n_1}{\sum_{i=1}^{n_1} log\left(1 + \frac{x_i}{\lambda}\right)} \tag{3.6}$$

$$\hat{\alpha_2}_{mle} = \frac{n_2}{\sum_{j=1}^{n_2} log\left(1 + \frac{y_j}{\lambda}\right)} \tag{3.7}$$

$$\hat{\alpha_3}_{mle} = \frac{n_3}{\sum_{k=1}^{n_3} log\left(1 + \frac{z_k}{\lambda}\right)} \tag{3.8}$$

Substituting this in (2.6) we obtained the MLE of R as

$$\hat{R}_{mle} = \frac{\hat{\alpha_1}\hat{\alpha_2}}{(\hat{\alpha_1} + \hat{\alpha_3})(\hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3})}; 0 < R < 1 \tag{3.9}$$

From the above expression, it is very difficult to find the exact variance and distribution of $\hat{R}_{mle}$. So we use the multivariate delta method (See [37], [38], [39], [40]) to find the approximate estimate of the asymptotic variance of $\hat{R}_{mle}$ which is given as

Let the Fisher Information matrix Ø

$$\text{Ø}\left(\alpha_1,\alpha_2,\alpha_3\right)=\begin{bmatrix} E\left(\frac{-\partial^2 lnL}{\partial\alpha_1^2}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_1\partial\alpha_2}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_1\partial\alpha_3}\right)\\ E\left(\frac{-\partial^2 lnL}{\partial\alpha_2\partial\alpha_1}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_2^2}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_2\partial\alpha_3}\right)\\ E\left(\frac{-\partial^2 lnL}{\partial\alpha_3\partial\alpha_1}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_3\partial\alpha_2}\right) & E\left(\frac{-\partial^2 lnL}{\partial\alpha_3^2}\right)\end{bmatrix} \tag{3.10}$$

$$B^{'}=\begin{bmatrix}\frac{\partial R}{\partial\alpha_1} & \frac{\partial R}{\partial\alpha_2} & \frac{\partial R}{\partial\alpha_3}\end{bmatrix}=\begin{bmatrix}b_1 & b_2 & b_3\end{bmatrix} \tag{3.11}$$

Then $\sigma_R^2=V\left(R\right)=B^{'}\text{Ø}^{-1}B$. In this case

$$\text{Ø}\left(\alpha_1,\alpha_2,\alpha_3\right)=\begin{bmatrix}\frac{n_1}{\alpha_1^2} & 0 & 0\\ 0 & \frac{n_2}{\alpha_2^2} & 0\\ 0 & 0 & \frac{n_3}{\alpha_3^2}\end{bmatrix}$$

So

$$\text{Ø}^{-1}=\begin{bmatrix}\frac{\alpha_1^2}{n_1} & 0 & 0\\ 0 & \frac{\alpha_2^2}{n_2} & 0\\ 0 & 0 & \frac{\alpha_3^2}{n_3}\end{bmatrix}$$

Also

$$b_1=\frac{\partial R}{\partial\alpha_1}=\frac{-\alpha_2\left(\alpha_1^2-\alpha_2\alpha_3-\alpha_2^2\right)}{\left(\alpha_1+\alpha_3\right)^2\left(\alpha_1+\alpha_2+\alpha_3\right)^2} \tag{3.12}$$

$$b_2=\frac{\partial R}{\partial\alpha_2}=\frac{\alpha_1}{\left(\alpha_1+\alpha_2+\alpha_3\right)^2} \tag{3.13}$$

and

$$b_3=\frac{\partial R}{\partial\alpha_3}=\frac{-\alpha_1\alpha_2(2\alpha_1+\alpha_2+2\alpha_3)}{\left(\alpha_1+\alpha_3\right)^2\left(\alpha_1+\alpha_2+\alpha_3\right)^2} \tag{3.14}$$

Then

$$\sigma_{Rmle}^2=V\left(R\right)=B^{'}\text{Ø}^{-1}B=\frac{b_1^2\alpha_1^2}{n_1}+\frac{b_2^2\alpha_2^2}{n_2}+\frac{b_3^2\alpha_3^2}{n_3} \tag{3.15}$$

By replacing the parameters with their maximum likelihood estimate we get the estimate $\hat{\sigma}_{Rmle}^2$ of σ_{Rmle}^2. In this case the asymptotic distribution of $\hat{R}_{mle}$ is $N\left(R,\hat{\sigma}_{Rmle}^2\right)$.

Based on this asymptotic distribution a $100\left(1-\gamma\right)\%$ asymptotic CI for R is $\hat{R}_{mle}\pm Z_{\gamma/2}\hat{\sigma}_{Rmle}$. Where $Z_{\gamma/2}$ denotes the table value corresponding to $\gamma/2$ of N(0,1).

3.1 Shrinkage Estimation with Constant Shrinkage Factor

In this case we obtain the shrinkage estimate,

$$\hat{\beta}_{sh}=\psi\left(\hat{\beta}\right)\hat{\beta}_{ub}+\left(1-\psi\left(\hat{\beta}\right)\right)\hat{\beta}_0$$

with $\psi\left(\hat{\beta}\right)=0.01$ the constant shrinkage weight factor suggested by [25] this leads the Shrinkage estimates of α_1, α_2 and α_3 as

$$\hat{\alpha_1}_{sh}=0.01\hat{\alpha_1}_{ub}+0.99\hat{\alpha_10} \tag{3.16}$$

$$\hat{\alpha_2}_{sh}=0.01\hat{\alpha_2}_{ub}+0.99\hat{\alpha_20} \tag{3.17}$$

and

$$\hat{\alpha_3}_{sh}=0.01\hat{\alpha_3}_{ub}+0.99\hat{\alpha_30} \tag{3.18}$$

where $\hat{\alpha 1}_{ub} = \frac{n_1-1}{n_1\bar{x}}$, $\hat{\alpha 2}_{ub} = \frac{n_2-1}{n_2\bar{y}}$ and $\hat{\alpha 3}_{ub} = \frac{n_3-1}{n_3\bar{z}}$. $\hat{\alpha 1_0}$, $\hat{\alpha 2_0}$ and $\hat{\alpha 3_0}$ is taken as the boot strap estimate of α_1, α_2 and α_3.

This leads to the constant shrinkage weight factor of R as

$$\hat{R}_{sh} = \frac{\hat{\alpha 1}_{sh}\hat{\alpha 2}_{sh}}{\left(\hat{\alpha 1}_{sh} + \hat{\alpha 2}_{sh} + \hat{\alpha 3}_{sh}\right)\left(\hat{\alpha 1}_{sh} + \hat{\alpha 3}_{sh}\right)} \tag{3.19}$$

3.2 The Modified Thompson Type Shrinkage Estimator

Here we use two type of shrinkage estimate first one the modified Thompson type shrinkage weight factor and Shrinkage weight factor suggested by [30] to find out the shrinkage estimator.
(a) Suggested by [25] here we take the weight factor as

$$\phi\left(\hat{R}\right) = \frac{\hat{R}_{ub} - \hat{R}_0}{\left(\hat{R}_{ub} - \hat{R}_0\right)^2 + var\left(\hat{R}_{ub}\right)}(0.001) \tag{3.20}$$

where $\hat{R}_{ub} = \frac{\hat{\alpha 1}_{ub}\hat{\alpha 2}_{ub}}{\left(\hat{\alpha 1}_{ub}+\hat{\alpha 2}_{ub}+\hat{\alpha 3}_{ub}\right)\left(\hat{\alpha 1}_{ub}+\hat{\alpha 3}_{ub}\right)}$ and $var\left(\hat{R}_{ub}\right)$ is as defined in (3.15). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Th} = \phi\left(\hat{R}\right)\hat{R}_{ub} + \left(1 - \phi\left(\hat{R}\right)\right)\hat{R}_0 \tag{3.21}$$

(b) Shrinkage weight factor suggested by [30] here we take the weight factor as

$$\varphi\left(\hat{R}\right) = a.exp\left\{-\frac{b\left(\hat{R}_{ub} - \hat{R}_0\right)^2}{var\left(\hat{R}_{ub}\right)}\right\} \tag{3.22}$$

where $0 < a < 1$ and $b > 0$. So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{MS} = \varphi\left(\hat{R}\right)\hat{R}_{ub} + \left(1 - \varphi\left(\hat{R}\right)\right)\hat{R}_0 \tag{3.23}$$

4 MAXIMUM LIKELIHOOD ESTIMATION OF R BASED ON RIGHT CENSORED SAMPLE

In this section we obtained the maximum likelihood estimate when the data on the stress is only is right censored. Let us consider a right censored sample $\underline{x} = \left(x_1, x_2, ..., x_{(n_1-k)}\right)$ with k observations censored on right from Lomax distribution $L(\alpha_1, \lambda)$ then its likelihood function is given by

$$L(\underline{x}|\alpha_1,\lambda) = \left[1 - F_{(n_1-k)}\right]^k \prod_{i=1}^{n_1-k} f(x_i) = \alpha_1^{(n_1-k)}\lambda^{(n_1-k)\alpha_1}\left[1 + \frac{x_{(n_1-k)}}{\lambda}\right]^{-\alpha_1 k}\prod_{i=1}^{n_1-k}\frac{1}{(x_i+\lambda)^{\alpha_1+1}} \tag{4.1}$$

Then using (3.2), (3.3) and (4.1) the joint likelihood function can be written as

$$\begin{aligned} L\left(\underline{x},\underline{y},\underline{z}|\alpha_1,\alpha_2,\alpha_3,\lambda\right) =& \alpha_1^{(n_1-k)}\lambda^{(n_1-k)\alpha_1}\left[1 + \frac{x_{(n_1-k)}}{\lambda}\right]^{-\alpha_1 k} \\ & \prod_{i=1}^{n_1-k}\frac{1}{(x_i+\lambda)^{\alpha_1+1}}\alpha_2^{n_2}\lambda^{n_2\alpha_2}\prod_{j=1}^{n_2}(y_j+\lambda)^{-(\alpha_2+1)}\alpha_3^{n_3}\lambda^{n_3\alpha_3}\times\prod_{k=1}^{n_3}(z_k+\lambda)^{-(\alpha_3+1)} \end{aligned} \tag{4.2}$$

Taking Logarithm on both side of (4.2) we get

$$\begin{aligned} logL = & -\alpha_1 k.log\left[1+\frac{x_{(n_1-k)}}{\lambda}\right]+(n_1-k)\,log\alpha_1+\alpha_1\,(n_1-k)\,log\lambda-(\alpha_1+1)\sum_{i=1}^{n_1-k}log\,(x_i+\lambda) \\ & +n_2 log\alpha_2+n_2\alpha_2 log\lambda-(\alpha_2+1)\sum_{j=1}^{n_2}log\,(y_j+\lambda) \\ & +n_3 log\alpha_3+n_3\alpha_3 log\lambda-(\alpha_3+1)\sum_{k=1}^{n_3}log\,(z_k+\lambda) \end{aligned} \quad (4.3)$$

From (4.3) we get the MLE of α_1, α_2 and α_3 as

$$\hat{\alpha}_{1mlec}=\frac{n_1-k}{\sum_{i=1}^{n_1-k}log\left(1+\frac{x_i}{\lambda}\right)+klog\left(1+\frac{x_{(n_1-k)}}{\lambda}\right)} \quad (4.4)$$

$$\hat{\alpha}_{2mlec}=\frac{n_2}{\sum_{j=1}^{n_2}log\left(1+\frac{y_j}{\lambda}\right)} \quad (4.5)$$

and

$$\hat{\alpha}_{3mlec}=\frac{n_3}{\sum_{k=1}^{n_3}log\left(1+\frac{z_k}{\lambda}\right)} \quad (4.6)$$

So using (4.4), (4.5) and (4.6) the MLE of R can be written as

$$\hat{R}_{mlec}=\frac{\hat{\alpha_1}\hat{\alpha_2}}{(\hat{\alpha_1}+\hat{\alpha_3})\,(\hat{\alpha_1}+\hat{\alpha_2}+\hat{\alpha_3})};0<R<1 \quad (4.7)$$

In this case

$$\varnothing^{-1}=\begin{bmatrix}\frac{\alpha_1^2}{n_1-k} & 0 & 0\\ 0 & \frac{\alpha_2^2}{n_2} & 0\\ 0 & 0 & \frac{\alpha_3^2}{n_3}\end{bmatrix} \quad (4.8)$$

Now using (3.12), (3.13), (3.14) and (4.8) we have

$$\sigma_{Rmlec}^2=V\,(R)=B^1\varnothing^{-1}B=\frac{b_1^2\alpha_1^2}{n_1-k}+\frac{b_2^2\alpha_2^2}{n_2}+\frac{b_3^2\alpha_3^2}{n_3} \quad (4.9)$$

By replacing the parameters with their maximum likelihood estimate we get the estimate $\hat{\sigma}_{Rmlec}^2$ of σ_{Rmlec}^2. In this case he asymptotic distribution of $\hat{R}_{mlec}$ is $N(R,\hat{\sigma}_{Rmlec}^2)$.Based on this asymptotic distribution a $100\,(1-\gamma)\,\%$ asymptotic CI for R is $\hat{R}_{mlec}\pm Z_{\gamma/2}\hat{\sigma}_{Rmlec}$.

4.1 Shrinkage Estimation with Constant Shrinkage Factor

In this case we obtain the shrinkage estimate, with $\psi\left(\hat{\beta}\right)=0.01$ the constant shrinkage weight factor leads the Shrinkage estimates of α_1, α_2 and α_3 as

$$\hat{\alpha 1}_{shc}=0.01\hat{\alpha 1}_{ub}+0.99\hat{\alpha 1}_1 \quad (4.10)$$

$$\hat{\alpha 2}_{shc}=0.01\hat{\alpha 2}_{ub}+0.99\hat{\alpha 2}_1 \quad (4.11)$$

and

$$\hat{\alpha 3}_{shc}=0.01\hat{\alpha 3}_{ub}+0.99\hat{\alpha 3}_1 \quad (4.12)$$

where $\hat{\alpha 1_{ub}} = \frac{n_1-1}{n_1\overline{x}}$, $\hat{\alpha 2_{ub}} = \frac{n_2-1}{n_2\overline{y}}$ and $\hat{\alpha 3_{ub}} = \frac{n_3-1}{n_3\overline{z}}$. $\hat{\alpha 1_1}$, $\hat{\alpha 2_1}$ and $\hat{\alpha 3_1}$ is taken as the boot strap estimate of $\alpha \hat{1_{mlec}}$, $\alpha \hat{2_{mlec}}$ and $\alpha \hat{3_{mlec}}$.

This leads to the constant shrinkage weight factor of R as

$$\hat{R}_{shc} = \frac{\alpha \hat{1_{shc}} \alpha \hat{2_{shc}}}{\left(\alpha \hat{1_{shc}} + \alpha \hat{2_{shc}} + \alpha \hat{3_{shc}}\right)\left(\alpha \hat{1_{shc}} + \alpha \hat{3_{shc}}\right)} \quad (4.13)$$

4.2 The Modified Thompson Type Shrinkage Estimator

The modified Thompson type shrinkage weight factor estimates suggested by [25] and [30] are

(a) $\phi\left(\hat{R}\right) = \frac{\hat{R}_{ub}-\hat{R}_{shc}}{\left(\hat{R}_{ub}-\hat{R}_{shc}\right)^2+var\left(\hat{R}_{ub}\right)}\,(0.001)$ (4.14)
where $\hat{R}_{ub} = \frac{\alpha \hat{1_{ub}} \alpha \hat{2_{ub}}}{\left(\alpha \hat{1_{ub}}+\alpha \hat{2_{ub}}+\alpha \hat{3_{ub}}\right)\left(\alpha \hat{1_{ub}}+\alpha \hat{3_{ub}}\right)}$ and $var\left(\hat{R}_{ub}\right)$ is as defined in (4.9). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Th} = \phi\left(\hat{R}\right)\hat{R}_{ub} + \left(1-\phi\left(\hat{R}\right)\right)\hat{R}_{shc} \quad (4.15)$$

(b) $\varphi\left(\hat{R}\right) = a.exp\left\{-\frac{b\left(\hat{R}_{ub}-\hat{R}_{shc}\right)^2}{var\left(\hat{R}_{ub}\right)}\right\}$ (4.16)
where $0 < a < 1$ and $b > 0$. So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{MS} = \varphi\left(\hat{R}\right)\hat{R}_{ub} + \left(1-\varphi\left(\hat{R}\right)\right)\hat{R}_{shc} \quad (4.17)$$

5 QUASI LIKELIHOOD ESTIMATION OF R BASED ON COMPLETE SAMPLE

In this section, we derived the maximum quasi-likelihood estimates for R. The quasi-likelihood function was introduced by [41] to be used for estimating the unknown parameters in generalized linear models when only the mean-variance relationship is specified. Wedderburn defined the quasi- function as

$$Q\left(x,\mu\right) = \int_{\mu} \frac{x-\mu}{V\left(\mu\right)}d\mu + o\left(x\right) \quad (5.1)$$

where $\mu = E\left(x\right)$, $V\left(\mu\right) = Var\left(x\right)$ and $o\left(x\right)$ is some function of x only. The variance assumption is generalized to $Var\left(x\right) = \phi V\left(\mu\right)$ where the variance function $V\left(.\right)$ is assumed to be known and the parameter ϕ may be unknown. The quasi-likelihood function has properties similar to those of the log-likelihood function. Let $\underline{x} = \left(x_1, x_2, ..., x_{n_1}\right)$ be the random sample of n_1 observation taken from Lomax distribution $L\left(\alpha_1, \lambda\right)$ then its Quasi Likelihood function is given by

$$Q\left(x_i, \alpha_1, \lambda\right) = n_1 log\left(\frac{\alpha_1-1}{\lambda}\right) - \nu\left(\frac{\alpha_1-1}{\lambda}\right) \quad (5.2)$$

where $\nu = \sum_{i=1}^{n_1} x_i$.

The natural exponent of $Q\left(x_i, \alpha_1, \lambda\right)$ as the as taken as the Quasi likelihood function and is given by

$$L\left(\underline{x}|\alpha_1, \lambda\right) = \left(\frac{\alpha_1-1}{\lambda}\right)^{n_1} e^{-\left(\frac{\alpha_1-1}{\lambda}\right)\nu}; \alpha_1 > 0, \nu = \sum_{i=1}^{n_1} x_i \quad (5.3)$$

Similar based on the sample $\underline{y} = \left(y_1, y_2, ..., y_{n_2}\right)$ and $\underline{z} = \left(z_1, z_2, ..., z_{n_3}\right)$ the Quasi likelihood function of Y and Z is given by

$$L\left(\underline{y}|\alpha_2, \lambda\right) = \left(\frac{\alpha_2-1}{\lambda}\right)^{n_2} e^{-\left(\frac{\alpha_2-1}{\lambda}\right)\zeta}; \alpha_2 > 0, \zeta = \sum_{j=1}^{n_2} y_j \quad (5.4)$$

and

$$L\left(\underline{z}|\alpha_3,\lambda\right)=\left(\frac{\alpha_3-1}{\lambda}\right)^{n_3}e^{-\left(\frac{\alpha_3-1}{\lambda}\right)\beta};\alpha_3>0,\beta=\sum_{k=1}^{n_3}z_k \tag{5.5}$$

So the joint quasi likelihood function can be written as

$$L\left(\underline{x},\underline{y},\underline{z}|\alpha_1,\alpha_2,\alpha_3,\lambda\right)=\left(\frac{\alpha_1-1}{\lambda}\right)^{n_1}e^{-\left(\frac{\alpha_1-1}{\lambda}\right)\nu}.\left(\frac{\alpha_2-1}{\lambda}\right)^{n_2}e^{-\left(\frac{\alpha_2-1}{\lambda}\right)\zeta}\left(\frac{\alpha_3-1}{\lambda}\right)^{n_3}e^{-\left(\frac{\alpha_3-1}{\lambda}\right)\beta} \tag{5.6}$$

From (5.6) the Quasi Likelihood Estimate of α_1, α_2, α_3 and R are obtained as

$$\hat{\alpha}_{1qmle}=1+\left(\frac{n_1\lambda}{\nu}\right)=1+\left(\frac{n_1\lambda}{\sum_{i=1}^{n_1}x_i}\right) \tag{5.7}$$

$$\hat{\alpha}_{2qmle}=1+\left(\frac{n_2\lambda}{\zeta}\right)=1+\left(\frac{n_2\lambda}{\sum_{j=1}^{n_2}y_j}\right) \tag{5.8}$$

$$\hat{\alpha}_{3qmle}=1+\left(\frac{n_3\lambda}{\beta}\right)=1+\left(\frac{n_3\lambda}{\sum_{k=1}^{n_3}z_k}\right) \tag{5.9}$$

and

$$\hat{R}_{qmle}=\frac{\hat{\alpha_1}\hat{\alpha_2}}{\left(\hat{\alpha_1}+\hat{\alpha_3}\right)\left(\hat{\alpha_1}+\hat{\alpha_2}+\hat{\alpha_3}\right)};0<R<1 \tag{5.10}$$

In this case

$$\varnothing^{-1}=\begin{bmatrix}\frac{(\alpha_1-1)^2}{n_1} & 0 & 0\\ 0 & \frac{(\alpha_2-1)^2}{n_1} & 0\\ 0 & 0 & \frac{(\alpha_3-1)^2}{n_3}\end{bmatrix} \tag{5.11}$$

Now using (3.12), (3.13), (3.14) and (5.11) we have

$$\sigma^2_{Rqmle}=V\left(R\right)=B^{'}\phi^{-1}B=\frac{b_1^2\left(\alpha_1-1\right)^2}{n_1}+\frac{b_2^2\left(\alpha_2-1\right)^2}{n_2}+\frac{b_3^2\left(\alpha_3-1\right)^2}{n_3} \tag{5.12}$$

By replacing the parameters with their maximum likelihood estimate we get the estimate $\hat{\sigma}^2_{Rqmle}$ of σ^2_{Rqmle}. In this case the asymptotic distribution of $\hat{R}_{qmle}$ is $N(R,\hat{\sigma}^2_{Rqmle})$. Based on this asymptotic distribution a $100\left(1-\gamma\right)\%$ asymptotic CI for R is $\hat{R}_{qmle}\pm Z_{\gamma/2}\hat{\sigma}_{Rqmle}$.

5.1 Shrinkage Estimates

In this case we obtain the different type of shrinkage estimates as,
(a) the constant weight shrinkage estimates with $\psi\left(\hat{\beta}\right)=0.01$ as

$$\hat{R}_{shq}=\frac{\hat{\alpha 1}_{shq}\hat{\alpha 2}_{shq}}{\left(\hat{\alpha 1}_{shq}+\hat{\alpha 2}_{shq}+\hat{\alpha 3}_{shq}\right)\left(\hat{\alpha 1}_{shq}+\hat{\alpha 3}_{shq}\right)} \tag{5.13}$$

$$\hat{\alpha 1}_{shq}=0.01\hat{\alpha 1}_{ub}+0.99\hat{\alpha 1}_2 \tag{5.14}$$

$$\hat{\alpha 2}_{shq}=0.01\hat{\alpha 2}_{ub}+0.99\hat{\alpha 2}_2 \tag{5.15}$$

and

$$\hat{\alpha 3}_{shq}=0.01\hat{\alpha 3}_{ub}+0.99\hat{\alpha 3}_2 \tag{5.16}$$

where $\hat{\alpha 1}_{ub}=\frac{n_1-1}{n_1\overline{x}}$, $\hat{\alpha 2}_{ub}=\frac{n_2-1}{n_2\overline{y}}$ and $\hat{\alpha 3}_{ub}=\frac{n_3-1}{n_3\overline{z}}$. $\hat{\alpha 1}_2$, $\hat{\alpha 2}_2$ and $\hat{\alpha 3}_2$ is taken as the boot strap estimate of $\hat{\alpha}_{1qmle}$, $\hat{\alpha}_{2qmle}$ and $\hat{\alpha}_{3qmle}$.

(b) Suggested by [25] here we take the weight factor as

$$\phi\left(\hat{R}\right)=\frac{\hat{R}_{ub}-\hat{R}_0}{\left(\hat{R}_{ub}-\hat{R}_{shq}\right)^2+var\left(\hat{R}_{ub}\right)}(0.001) \tag{5.17}$$

where $\hat{R}_{ub}=\frac{\hat{\alpha 1_{ub}}\hat{\alpha 2_{ub}}}{\left(\hat{\alpha 1_{ub}}+\hat{\alpha 2_{ub}}+\hat{\alpha 3_{ub}}\right)\left(\hat{\alpha 1_{ub}}+\hat{\alpha 3_{ub}}\right)}$ and $var\left(\hat{R}_{ub}\right)$ is as defined in (5.12). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Thq}=\phi\left(\hat{R}\right)\hat{R}_{ub}+\left(1-\phi\left(\hat{R}\right)\right)\hat{R}_{shq} \tag{5.18}$$

(c) Shrinkage weight factor suggested by [30] here we take the weight factor as

$$\varphi\left(\hat{R}\right)=a.exp\left\{-\frac{b\left(\hat{R}_{ub}-\hat{R}_{shq}\right)^2}{var\left(\hat{R}_{ub}\right)}\right\} \tag{5.19}$$

where $0<a<1$ and $b>0$. So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{MS}=\varphi\left(\hat{R}\right)\hat{R}_{ub}+\left(1-\varphi\left(\hat{R}\right)\right)\hat{R}_{shq} \tag{5.20}$$

6 QUASI LIKELIHOOD ESTIMATION OF R BASED ON RIGHT CENSORED SAMPLE

As in the case of section 4 in this case also we considered a right censoring procedure. Let $\underline{x}=(x_1,x_2,...,x_{n_1-k})$ be the random sample of (n_1-k) observation taken from Lomax distribution $L(\alpha_1,\lambda)$ then its Quasi function is given by

$$Q(x_i,\alpha_1,\lambda)=(n_1-k)\,log\left(\frac{\alpha_1-1}{\lambda}\right)-\nu\left(\frac{\alpha_1-1}{\lambda}\right) \tag{6.1}$$

where $\nu=\sum_{i=1}^{n_1-k}x_i$. So the quasi likelihood function in this case is given as

$$L(\underline{x}|\alpha_1,\lambda)=\left(\frac{\alpha_1-1}{\lambda}\right)^{n_1}e^{-\left(\frac{\alpha_1-1}{\lambda}\right)\nu};\alpha_1>0,\nu=\sum_{i=1}^{n_1-k}x_i \tag{6.2}$$

Now using (6.2), (5.4) an (5.5) the joint likelihood function can be written as

$$L\left(\underline{x},\underline{y},\underline{z}|\alpha_1,\alpha_2,\alpha_3,\lambda\right)=\left(\frac{\alpha_1-1}{\lambda}\right)^{n_1-k}e^{-\left(\frac{\alpha_1-1}{\lambda}\right)\nu}.\left(\frac{\alpha_2-1}{\lambda}\right)^{n_2}e^{-\left(\frac{\alpha_2-1}{\lambda}\right)\zeta}.\left(\frac{\alpha_3-1}{\lambda}\right)^{n_3}e^{-\left(\frac{\alpha_3-1}{\lambda}\right)\beta} \tag{6.3}$$

Using (6.3) we get the estimate of α_1, α_2, α_3 and R are obtained as

$$\hat{\alpha}_{1qmlec}=1+\left(\frac{(n_1-k)\,\lambda}{\nu}\right)=1+\left(\frac{(n_1-k)\,\lambda}{\sum_{i=1}^{(n_1-k)}x_i}\right) \tag{6.4}$$

$$\hat{\alpha}_{2qmlec}=1+\left(\frac{n_2\lambda}{\zeta}\right)=1+\left(\frac{n_2\lambda}{\sum_{j=1}^{n_2}y_j}\right) \tag{6.5}$$

$$\hat{\alpha}_{3qmlec} = 1 + \left(\frac{n_3\lambda}{\beta}\right) = 1 + \left(\frac{n_3\lambda}{\sum_{k=1}^{n_3} z_k}\right) \tag{6.6}$$

and

$$\hat{R}_{qmlec} = \frac{\hat{\alpha_1}\hat{\alpha_2}}{(\hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3})(\hat{\alpha_1} + \hat{\alpha_3})}; 0 < R < 1 \tag{6.7}$$

In this case

$$\varnothing^{-1} = \begin{bmatrix} \frac{(\alpha_1-1)^2}{n_1-k} & 0 & 0 \\ 0 & \frac{(\alpha_2-1)^2}{n_1} & 0 \\ 0 & 0 & \frac{(\alpha_3-1)^2}{n_3} \end{bmatrix} \tag{6.8}$$

Now using (3.12), (3.13), (3.14) and (6.8) we have

$$\sigma^2_{Rqmlec} = V(R) = B^{'}\phi^{-1}B = \frac{b_1^2(\alpha_1-1)^2}{n_1-k} + \frac{b_2^2(\alpha_2-1)^2}{n_2} + \frac{b_3^2(\alpha_3-1)^2}{n_3} \tag{6.9}$$

By replacing the parameters with their maximum likelihood estimate we get the estimate $\hat{\sigma}^2_{Rqmlec}$ of σ^2_{Rqmlec}. In this case he asymptotic distribution of $\hat{R}_{qmlec}$ is $N(R, \hat{\sigma}^2_{Rqmlec})$. Based on this asymptotic distribution a $100(1-\gamma)\%$ asymptotic CI for R is $\hat{R}_{qmlec} \pm Z_{\gamma/2}\hat{\sigma}_{Rqmlec}$.

6.1 Shrinkage Estimates

In this case we obtain the shrinkage estimate, with $\psi\left(\hat{\beta}\right) = 0.01$ the constant shrinkage weight factor suggested by [25].

This leads to the constant shrinkage weight factor of R as $\hat{R}_{shqc}$

$$\hat{R}_{shqc} = \frac{\alpha\hat{1}_{shqc}\alpha\hat{2}_{shqc}}{\left(\alpha\hat{1}_{shqc} + \alpha\hat{2}_{shqc} + \alpha\hat{3}_{shqc}\right)\left(\alpha\hat{1}_{shqc} + \alpha\hat{3}_{shqc}\right)} \tag{6.10}$$

with

$$\alpha\hat{1}_{shqc} = 0.01\alpha\hat{1}_{ub} + 0.99\alpha\hat{1}3 \tag{6.11}$$

$$\alpha\hat{2}_{shqc} = 0.01\alpha\hat{2}_{ub} + 0.99\alpha\hat{2}3 \tag{6.12}$$

and

$$\alpha\hat{3}_{shqc} = 0.01\alpha\hat{3}_{ub} + 0.99\alpha\hat{3}3 \tag{6.13}$$

where $\alpha\hat{1}_{ub} = \frac{n_1-1}{n_1\bar{x}}$, $\alpha\hat{2}_{ub} = \frac{n_2-1}{n_2\bar{y}}$ and $\alpha\hat{3}_{ub} = \frac{n_3-1}{n_3\bar{z}}$. $\alpha\hat{1}3$, $\alpha\hat{2}3$ and $\alpha\hat{3}3$ is taken as the boot strap estimate of $\alpha\hat{1}_{qmlec}$, $\alpha\hat{2}_{qmlec}$ and $\alpha\hat{3}_{qmlec}$.

Also the modified Thompson type shrinkage weight factor and Shrinkage estimate by [30] are

(b) Suggested by [25] here we take the weight factor as

$$\phi\left(\hat{R}\right) = \frac{\hat{R}_{ub} - \hat{R}_{shqc}}{\left(\hat{R}_{ub} - \hat{R}_{shqc}\right)^2 + var\left(\hat{R}_{ub}\right)}(0.001) \tag{6.14}$$

where $\hat{R}_{ub} = \frac{\hat{\alpha 1}_{ub}\hat{\alpha 2}_{ub}}{\left(\hat{\alpha 1}_{ub}+\hat{\alpha 2}_{ub}+\hat{\alpha 3}_{ub}\right)\left(\hat{\alpha 1}_{ub}+\hat{\alpha 3}_{ub}\right)}$ and $var\left(\hat{R}_{ub}\right)$ is as defined in (6.9). So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{Th} = \phi\left(\hat{R}\right)\hat{R}_{ub} + \left(1 - \phi\left(\hat{R}\right)\right)\hat{R}_{shqc} \quad (6.15)$$

(c) Shrinkage weight factor suggested by [30] here we take the weight factor as

$$\varphi\left(\hat{R}\right) = a.exp\left\{-\frac{b\left(\hat{R}_{ub} - \hat{R}_{shqc}\right)^2}{var\left(\hat{R}_{ub}\right)}\right\} \quad (6.16)$$

where $0 < a < 1$ and $b > 0$. So the modified Thomason type shrinkage estimator will be

$$\hat{R}_{MS} = \varphi\left(\hat{R}\right)\hat{R}_{ub} + \left(1 - \varphi\left(\hat{R}\right)\right)\hat{R}_{shqc} \quad (6.17)$$

7 SIMULATION STUDY

In this section we obtained the numerical results using simulation data.

Here, we have considered a bootstrap CI for r by using a parametric percentile bootstrap method ([42]). The following algorithm is used to generate the parametric bootstrap estimates of R.

Step-1. Simulate a random sample from Uniform (0,1). Using this simulated value compute random sample for $X \sim L\left(\alpha_1, \lambda\right)$, $Y \sim L\left(\alpha_2, \lambda\right)$ and $Z \sim L\left(\alpha_3, \lambda\right)$ respectively.

Compute the MLE of α_1, α_2, α_3 say $\hat{\alpha}_{1mle}$, $\hat{\alpha}_{2mle}$, $\hat{\alpha}_{3mle}$ given in setion-2.

Step-2. Generate an independent parametric bootstrap sample using $\hat{\alpha}_{1mle}$, $\hat{\alpha}_{2mle}$, $\hat{\alpha}_{3mle}$ instead of α_1, α_2, α_3. Then using these values, calculate $\hat{R}_{mle}$ in the case of complete sample. Similar way generates an independent parametric bootstrap sample with a censoring of 30% and 50%. using $\hat{\alpha}_{1mlec}$, $\hat{\alpha}_{2mlec}$, $\hat{\alpha}_{3mlec}$ instead of α_1, α_2, α_3. Then using these values, calculate $\hat{R}_{mlec}$.

Step-3. Calculate the maximum likelihood estimate of $\hat{\alpha}_{1mle}$, $\hat{\alpha}_{2mle}$, $\hat{\alpha}_{3mle}$ and $\hat{R}_{mle}$ obtained in step-2 say $\hat{\alpha'}_{1mle}$, $\hat{\alpha'}_{2mle}$, $\hat{\alpha'}_{3mle}$ and $\hat{R'}_{1mle}$. In censored sample , the maximum likelihood estimate of $\hat{\alpha}_{1mlec}$, $\hat{\alpha}_{2mlec}$, $\hat{\alpha}_{3mlec}$ and $\hat{R}_{mlec}$ obtained in step-2 say $\hat{\alpha'}_{1mlec}$, $\hat{\alpha'}_{2mlec}$, $\hat{\alpha'}_{3mlec}$ and $\hat{R'}_{mlec}$.

Step-4. Repeat the step-2 and step-3 N times to obtained the parametric bootstrap estimates $\hat{R'}_{ML1}, \hat{R'}_{ML2}, ..., \hat{R'}_{MLN}$ of R.

Step-5. Let $H\left(x\right) = P\left(\hat{R}_{ML} \leq x\right)$ be the cumulative distribution function of $\hat{R}_{ML}$. Define $\hat{R}_{Boot}\left(x\right) = H^{-1}\left(x\right)$ for a given x. The approximate $100\left(1-\gamma\right)\%$ CI of R is given by
$\left(\hat{R}_{Boot}\left(\gamma/2\right), \hat{R}_{Boot}\left(1-\gamma/2\right)\right)$.

In the absence of real data finally, we study the performance of the estimates obtained in the above section using Monte Carlo Simulated data sets. All the computations are done by using R Program.

Generate the sample of sizes $n_1 = n_2, n_3 = (10, 10), (10, 25), (10, 50), (25, 10), (25, 25), (25, 50), (50, 10), (50, 25), (50, 50)$ from Lomax Distribution with parameter values 0.5,2,3.5 for α_1, α_2 and α_3. The bias, mean square error, confidence interval and relative efficiency are calculated and are given in the following table.

Table1: Bias, MSE and Relative Efficiency of the estimates of Reliability functions under complete sample.

$n_1=n_2$	n_3	$\alpha_1 = \alpha_2$	α_3		RMLE	Rsh	RTh	RMs
				Bias	0.0278	0.02201	0.0276	0.02552
10	10			MSE	0.0046	0.00139	0.0045	0.00311
				RE		69.78261	2.17391	32.3913
				Bias	0.0355	0.0067	0.0354	0.0199
25	10	0.5	0.5	MSE	0.0032	0.00011	0.0012	0.0001
				RE		96.5625	62.5	96.875
				Bias	0.0127	0.00842	0.0156	0.0229
50	10			MSE	0.0013	0.0005	0.00125	0.00107
				RE		61.53846	3.84615	17.6923
				Bias	0.006	0.00186	0.00595	0.00371
10	25			MSE	0.0004	0.00018	0.000348	0.00028
				RE		55	13	30
				Bias	0.0077	0.00598	0.0067	0.0087
25	25			MSE	0.0003	0.00011	0.00023	0.00027
				RE		63.33333	23.33333	10
		0.5	2	Bias	0.0025	0.0022	0.00249	0.00181
				MSE	0.00013	0.0001	0.00012	0.00011
50	25			RE		23.07692	7.69230	15.38461
				Bias	0.0063	0.00618	0.00631	0.00228
10	50			MSE	0.0002	0.00014	0.00019	0.00012
				RE		30	5	40
				Bias	0.0108	0.00562	0.01079	0.04648
25	50	0.5	3.5	MSE	0.0004	0.00012	0.00038	0.00024
				RE		70	5	40
				Bias	0.0081	0.00408	0.00813	0.00432
50	50			MSE	0.0005	0.0001	0.00046	0.000108
				RE		80	8	78.4
				Bias	0.0208	0.01786	0.0201	0.01714
10	10			MSE	0.0088	0.00654	0.0086	0.00719
				RE		25.68182	2.27272	18.29545
		2	0.5	Bias	0.0418	0.09197	0.04124	0.0123
25	10			MSE	0.0043	0.0024	0.00424	0.00204
				RE		44.18605	1.39534	52.55813

50	10			Bias	0.0013	0.00154	0.00177	0.00119
				MSE	0.0048	0.00395	0.0046	0.00372
				RE		17.70833	4.16666	22.5
10	25			Bias	0.0064	0.00288	0.00602	0.00117
				MSE	0.002	0.00163	0.00193	0.00125
				RE		18.5	3.5	37.5
25	25			Bias	0.0147	0.0191	0.0146	0.01129
				MSE	0.0021	0.001	0.0011	0.001
				RE		52.38095	47.61904	52.3809
50	25	2	2	Bias	0.059	0.02413	0.02413	0.02283
				MSE	0.0184	0.00105	0.01489	0.0108
				RE		94.29348	19.07608	41.30434
10	50			Bias	0.0189	0.01958	0.0184	0.01204
				MSE	0.0022	0.00106	0.00214	0.00146
				RE		51.81818	2.72727	33.63636
25	50	2	3.5	Bias	0.0628	0.04496	0.02156	0.01164
				MSE	0.00089	0.00017	0.0006	0.00025
				RE		80.89888	32.58426	71.91011
50	50			Bias	0.0008	0.00046	0.00074	0.00067
				MSE	0.00067	0.0001	0.00052	0.00037
				RE		85.07463	22.3880	44.77611
10	10			Bias	0.0634	0.08927	0.06215	0.01124
				MSE	0.00985	0.00121	0.00201	0.00144
				RE		87.71574	79.59390	85.380710
25	10	3.5	0.5	Bias	0.0033	0.0055	0.0034	0.00645
				MSE	0.00178	0.0011	0.00122	0.0012
				RE		38.20225	31.4606	32.58426
50	10			Bias	0.0105	0.01538	0.01061	0.01269
				MSE	0.0022	0.00198	0.002	0.00197
				RE		10	9.0909	10.45454
10	25			Bias	0.05422	0.045	0.04406	0.03653
				MSE	0.00708	0.0026	0.00356	0.00277
				RE		63.27684	49.7175	60.87570
25	25	3.5	2	Bias	0.0101	0.00509	0.01171	0.00998
				MSE	0.00564	0.0022	0.00445	0.00243
				RE		60.99291	21.09929	56.9148
50	25			Bias	0.0213	0.01085	0.021	0.0119
				MSE	0.0025	0.00179	0.0024	0.002
				RE		28.4	4	20

10	50			Bias	0.0123	0.01168	0.01266	0.01238
				MSE	0.0027	0.00153	0.00262	0.0018
				RE		43.33333	2.9629	33.3333
25	50	3.5	3.5	Bias	0.08267	0.07287	0.07861	0.0742
				MSE	0.008	0.00173	0.0072	0.00394
				RE		78.375	10	50.75
50	50			Bias	0.0903	0.01578	0.08922	0.04195
				MSE	0.008	0.00165	0.00607	0.00229
				RE		79.375	24.125	71.375

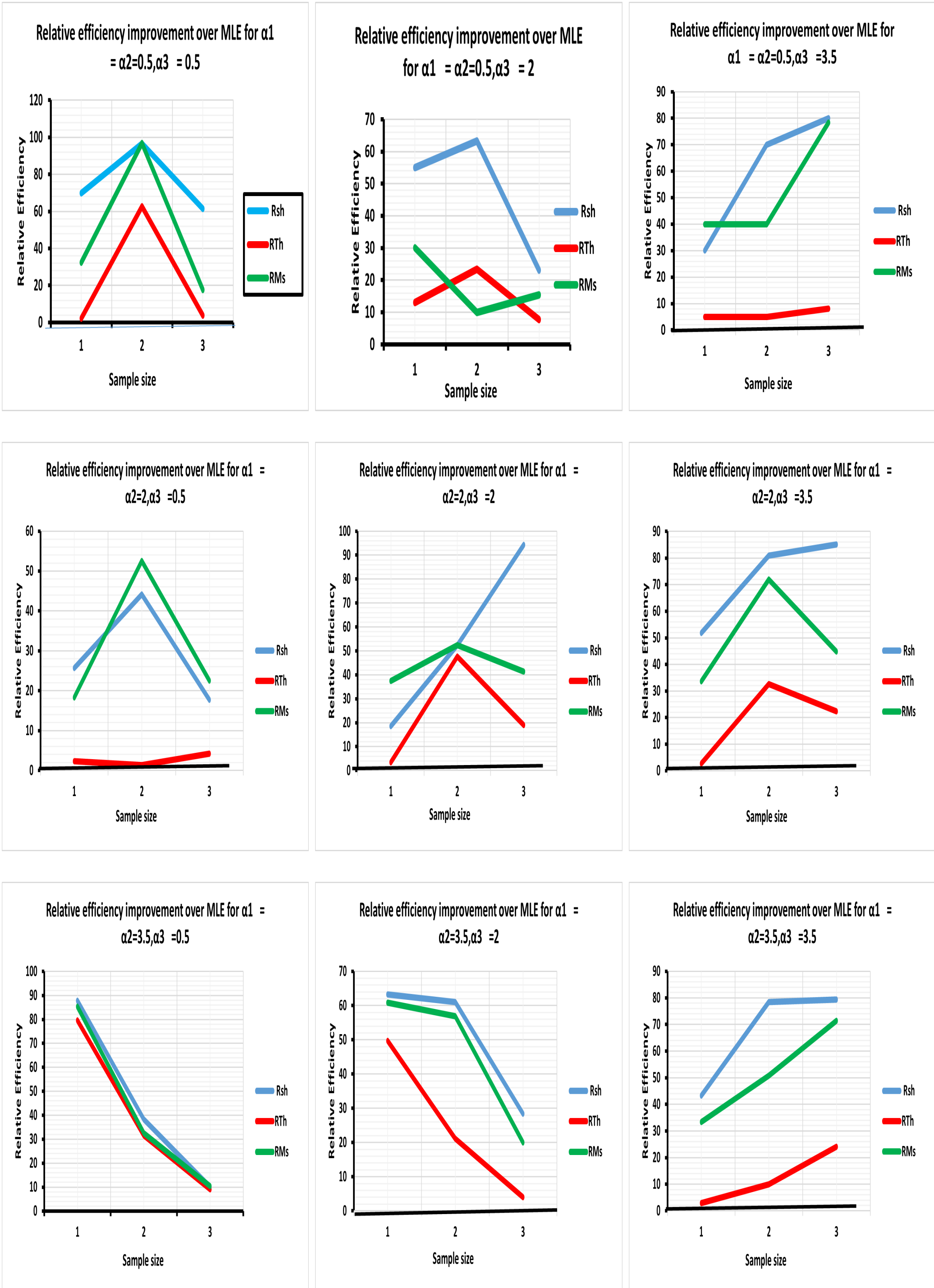

Fig. 1. Relative efficiency improvement Over MLE with complete sample

Table2:Bias, MSE and Relative Efficiency of the estimates of Reliability functions under censored sample(30% censoring).

$n_1=n_2$	n_3	$\alpha_1 = \alpha_2$	α_3		RMLE	Rsh	RTh	RMs
10				Bias	0.017598	0.01263	0.01531	0.01381
	10			MSE	0.009602	0.00161	0.00526	0.00423
				RE		83.23278	45.22013	55.94699
				Bias	0.07503	0.00153	0.00456	0.00193
25	10	0.5	0.5	MSE	0.00616	0.00014	0.000395	0.00025
				RE		97.72727	93.58766	95.94156
				Bias	0.0206	0.012184	0.03241	0.015582
	10			MSE	0.00343	0.00116	0.00227	0.00151
50				RE		66.18076	33.81924	55.97668
				Bias	0.03282	0.01567	0.0233	0.0166
10	25			MSE	0.00183	0.0011	0.00174	0.00142
				RE		39.89071	4.918033	22.40437
				Bias	0.01048	0.01124	0.0735	0.01604
25	25	0.5	2	MSE	0.00299	0.00101	0.00214	0.00135
				RE		66.22074	28.42809	54.8495
				Bias	0.02394	0.01809	0.03106	0.02208
50	25			MSE	0.00162	0.0002	0.00057	0.00041
				RE		87.65432	64.81481	74.69136
				Bias	0.03163	0.00115	0.00796	0.00254
10	50			MSE	0.002131	0.000128	0.00048	0.00027
				RE		93.99343	77.47536	87.32989
				Bias	0.03551	0.01392	0.01751	0.01697
25	50	0.5	3.5	MSE	0.00195	0.00056	0.00065	0.0006
				RE		71.28205	66.66667	69.23077
				Bias	0.10674	0.001256	0.001851	0.001285
50	50			MSE	0.03475	0.00012	0.000179	0.000163
				RE		99.65468	99.48489	99.53094
				Bias	0.06811	0.01574	0.07567	0.0651
10	10			MSE	0.00532	0.00109	0.00133	0.00123
				RE		79.51128	75	76.8797
				Bias	0.01283	0.01185	0.01261	0.013374
25	10	2	0.5	MSE	0.00632	0.00271	0.0053	0.00513
				RE		57.12025	16.13924	18.82911
				Bias	0.13262	0.018195	0.08456	0.0529
50	10			MSE	0.01805	0.001138	0.014294	0.01393
				RE		93.69529	20.80886	22.82548

				Bias	0.09171	0.011242	0.09075	0.04584
10	25			MSE	0.00855	0.00232	0.00837	0.00436
				RE		72.8655	2.105263	49.00585
				Bias	0.04352	0.0112	0.04312	0.04088
25	25	2	2	MSE	0.00418	0.00239	0.00406	0.00312
				RE		42.82297	2.870813	25.35885
				Bias	0.0632	0.01934	0.05783	0.02259
50	25			MSE	0.00505	0.00113	0.00441	0.00383
				RE		77.62376	12.67327	24.15842
				Bias	0.05729	0.01227	0.05679	0.02656
10	50			MSE	0.00461	0.00101	0.00452	0.00121
				RE		78.09111	1.952278	73.75271
				Bias	0.06276	0.01347	0.0613	0.01415
25	50	2	3.5	MSE	0.01063	0.001009	0.00127	0.00121
				RE		90.508	88.05268	88.61712
				Bias	0.025	0.013057	0.02405	0.017793
50	50			MSE	0.0047	0.0006	0.00127	0.00116
				RE		87.23404	72.97872	75.31915
				Bias	0.07553	0.01281	0.02035	0.02081
10	10			MSE	0.0237	0.00196	0.00442	0.00431
				RE		91.72996	81.35021	81.81435
				Bias	0.02163	0.06671	0.0858	0.06477
25	10	3.5	0.5	MSE	0.00747	0.00251	0.00693	0.0055
				RE		66.39893	7.228916	26.37216
				Bias	0.04528	0.01283	0.015907	0.01491
50	10			MSE	0.01668	0.00237	0.00538	0.00475
				RE		85.79137	67.7458	71.52278
				Bias	0.051	0.02156	0.02411	0.02368
10	25			MSE	0.00547	0.00212	0.00331	0.00329
				RE		61.24314	39.48812	39.85375
				Bias	0.08241	0.07537	0.05772	0.06982
25	25	3.5	2	MSE	0.00554	0.00172	0.00489	0.00218
				RE		68.95307	11.73285	60.64982
				Bias	0.03324	0.01372	0.0244	0.02392
50	25			MSE	0.01282	0.00215	0.00735	0.00636
				RE		83.22933	42.66771	50.39002
				Bias	0.0361	0.02413	0.0333	0.024654
10	50			MSE	0.0095	0.00402	0.00508	0.00471
		3.5	3.5	RE		57.68421	46.52632	50.42105
25	50			Bias	0.06619	0.01952	0.02157	0.02107
				MSE	0.01878	0.00127	0.0131	0.01
				RE		93.23749	30.24494	46.75186
				Bias	0.05692	0.0376	0.05133	0.0428
50	50			MSE	0.01199	0.00125	0.00906	0.007737
				RE		89.57465	24.43703	35.47123

Table3: Bias, MSE and Relative Efficiency of the estimates of Reliability functions under censored sample(50% censoring).

$n_1=n_2$	n_3	$\alpha_1=\alpha_2$	α_3		RMLE	Rsh	RTh	RMs
10	10	0.5	0.5	Bias	0.01305	0.01263	0.01531	0.01381
				MSE	0.00739	0.00161	0.00526	0.00423
				RE		78.2138	28.82273	42.76049
25	10			Bias	0.06644	0.00153	0.00456	0.00193
				MSE	0.00532	0.00014	0.000395	0.00025
				RE		97.36842	92.57519	95.30075
50	10			Bias	0.04486	0.012184	0.03241	0.015582
				MSE	0.00351	0.00116	0.00227	0.00151
				RE		66.95157	35.32764	56.98006
10	25	0.5	2	Bias	0.0221	0.01567	0.0233	0.0166
				MSE	0.003112	0.0011	0.00174	0.00142
				RE		64.65296	44.0874	54.37018
25	25			Bias	0.01827	0.01124	0.0735	0.01604
				MSE	0.0036	0.00101	0.00214	0.00135
				RE		71.94444	40.55556	62.5
50	25			Bias	0.0962	0.01809	0.03106	0.02208
				MSE	0.00084	0.0002	0.00057	0.00041
				RE		76.19048	32.14286	51.19048
10	50	0.5	3.5	Bias	0.01371	0.00115	0.00796	0.00254
				MSE	0.00207	0.000128	0.00048	0.00027
				RE		93.81643	76.81159	86.95652
25	50			Bias	0.01196	0.01392	0.01751	0.01697
				MSE	0.00075	0.00056	0.00065	0.0006
				RE		25.33333	13.33333	20
50	50			Bias	0.15391	0.001256	0.001851	0.001285
				MSE	0.08619	0.00012	0.000179	0.000163
				RE		99.86077	99.79232	99.81088
10	10	2	0.5	Bias	0.06503	0.01574	0.07567	0.0651
				MSE	0.01937	0.00109	0.00133	0.00123
				RE		94.37274	93.13371	93.64997
25	10			Bias	0.05112	0.01185	0.01261	0.013374
				MSE	0.00656	0.00271	0.0053	0.00513
				RE		58.68902	19.20732	21.79878

50	10			Bias	0.14913	0.018195	0.08456	0.0529
				MSE	0.02435	0.001138	0.014294	0.01393
				RE		95.32649	41.29774	42.79261
10	25			Bias	0.07639	0.011242	0.09075	0.04584
				MSE	0.00907	0.00232	0.00837	0.00436
				RE		74.42117	7.717751	51.92944
25	25	2	2	Bias	0.03659	0.0112	0.04312	0.04088
				MSE	0.00857	0.00239	0.00406	0.00312
				RE		72.11202	52.62544	63.59393
50	25			Bias	0.09204	0.01934	0.05783	0.02259
				MSE	0.02367	0.00113	0.00441	0.00383
				RE		95.22602	81.36882	83.81918
10	50			Bias	0.11388	0.01227	0.05679	0.02656
				MSE	0.01354	0.00101	0.00452	0.00121
				RE		92.54062	66.61743	91.06352
25	50	2	3.5	Bias	0.04191	0.01347	0.0613	0.01415
				MSE	0.00786	0.001009	0.00127	0.00121
				RE		87.16285	83.84224	84.6056
50	50			Bias	0.02663	0.013057	0.02405	0.017793
				MSE	0.00162	0.0006	0.00127	0.00116
				RE		62.96296	21.60494	28.39506
10	10			Bias	0.03797	0.01281	0.02035	0.02081
				MSE	0.00788	0.00196	0.00442	0.00431
				RE		75.1269	43.90863	45.30457
25	10	3.5	0.5	Bias	0.04847	0.06671	0.0858	0.06477
				MSE	0.00715	0.00251	0.00693	0.0055
				RE		64.8951	3.076923	23.07692
50	10			Bias	0.02569	0.01283	0.015907	0.01491
				MSE	0.00615	0.00237	0.00538	0.00475
				RE		61.46341	12.52033	22.76423
10	25			Bias	0.08	0.02156	0.02411	0.02368
				MSE	0.00769	0.00212	0.00331	0.00329
		3.5	2	RE		72.43173	56.95709	57.21717
25	25			Bias	0.07553	0.07537	0.05772	0.06982
				MSE	0.00542	0.00172	0.00489	0.00218
				RE		68.26568	9.778598	59.7786
50	25			Bias	0.09709	0.01372	0.0244	0.02392
				MSE	0.03503	0.00215	0.00735	0.00636
				RE		93.8624	79.01798	81.84413
10	50	3.5	3.5	Bias	0.02498	0.02413	0.0333	0.024654
				MSE	0.00604	0.00402	0.00508	0.00471
				RE		33.44371	15.89404	22.01987
25	50			Bias	0.06869	0.01952	0.02157	0.02107
				MSE	0.01911	0.00127	0.0131	0.01
				RE		93.35426	31.4495	47.67138
50	50			Bias	0.07968	0.0376	0.05133	0.0428
				MSE	0.01994	0.00125	0.00906	0.007737
				RE		93.73119	54.56369	61.1986

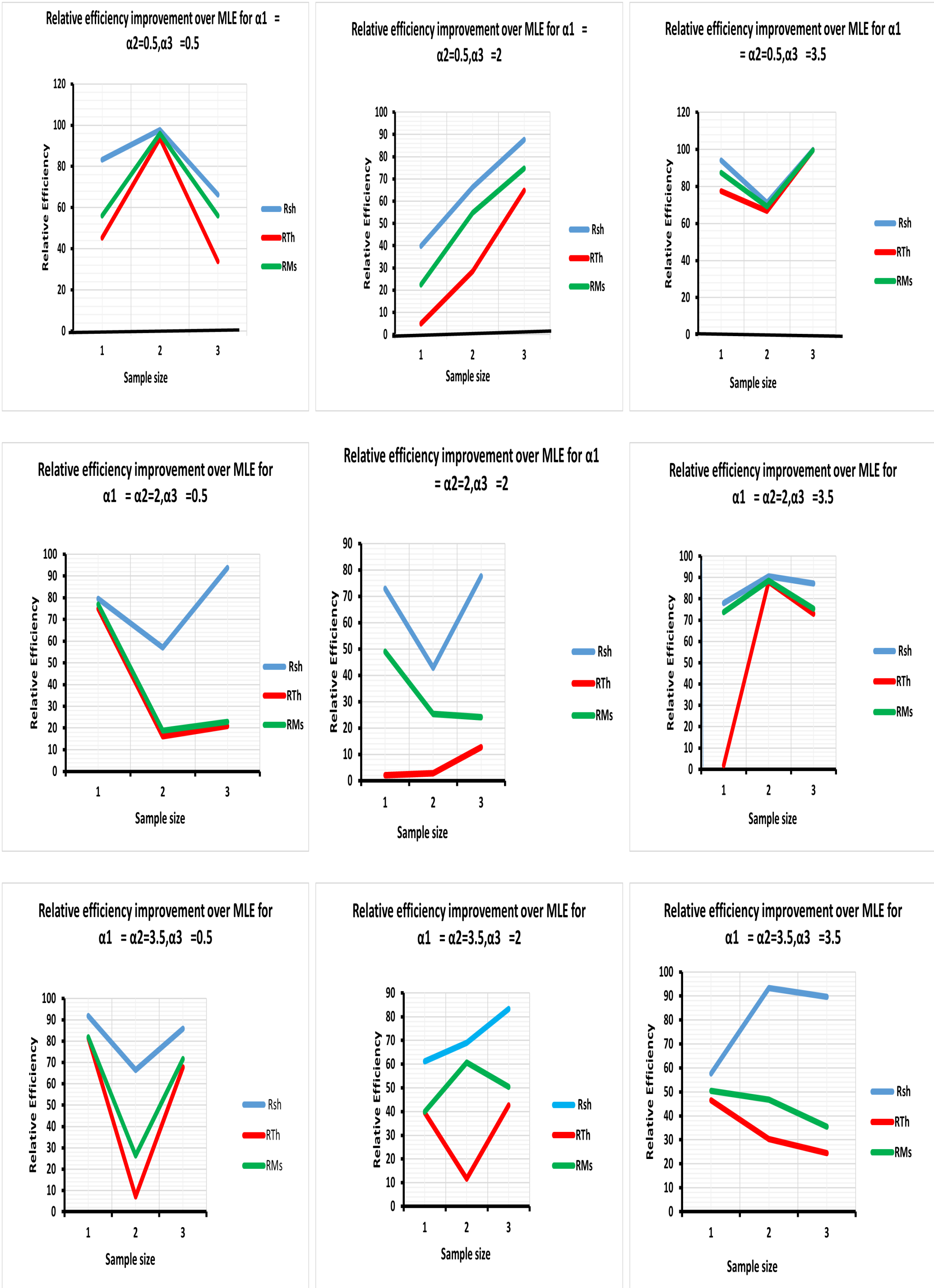

Fig. 2. Relative efficiency improvement Over MLE with censored sample

Table4:Bias, MSE and Relative Efficiency of the estimates of Reliability functions under complete sample using Quasi Likelihood Estimation

$n_1=n_2$	n_3	$\alpha_1 = \alpha_2$	α_3		RMLE	Rsh	RTh	RMs
10	10	0.5	0.5	Bias	0.018288	0.01006	0.06929	0.021592
				MSE	0.001748	0.00104	0.00147	0.00113
				RE		40.04576	15.44622	34.89703
25	10			Bias	0.01599	0.01021	0.01492	0.011712
				MSE	0.0018	0.001	0.00107	0.00102
				RE		44.44444	40.55556	43.33333
50	10			Bias	0.0861	0.03263	0.0834	0.0536
				MSE	0.0022	0.00102	0.00133	0.00119
				RE		53.63636	39.54545	45.90909
10	25	0.5	2	Bias	0.0588	0.01147	0.05024	0.04383
				MSE	0.00461	0.00141	0.00308	0.00102
				RE		69.41431	33.18872	77.87419
25	25			Bias	0.08438	0.04895	0.08139	0.079
				MSE	0.00776	0.00303	0.0043	0.00399
				RE		60.95360	44.58763	48.58247
50	25			Bias	0.0595	0.01476	0.05801	0.05526
				MSE	0.0042	0.00232	0.00392	0.00356
				RE		44.76190	6.6666	15.2381
10	50	0.5	3.5	Bias	0.0812	0.0201	0.0381	0.0311
				MSE	0.00611	0.00154	0.00231	0.0021
				RE		74.7954173	62.19313	65.63011
25	50			Bias	0.03186	0.02046	0.02533	0.02066
				MSE	0.00226	0.00051	0.00114	0.00094
				RE		77.43362	49.55752	58.40708
50	50			Bias	0.02969	0.01624	0.02288	0.01866
				MSE	0.00112	0.0001	0.00016	0.00014
				RE		91.07142	85.71429	87.5
10	10	2	0.5	Bias	0.0813	0.01682	0.08111	0.02348
				MSE	0.001107	0.00014	0.00109	0.00046
				RE		87.35320	1.53568	58.44625
25	10			Bias	0.08198	0.01061	0.017925	0.01079
				MSE	0.00771	0.00126	0.004517	0.00222
				RE		83.65758	41.41375	71.20623

50	10			Bias	0.08511	0.0125	0.05476	0.013106
				MSE	0.00735	0.00393	0.00529	0.0041
				RE		46.53061	28.02721	44.21769
10	25			Bias	0.04585	0.02381	0.036282	0.02883
				MSE	0.00463	0.00114	0.00259	0.00174
				RE		75.37796	44.06048	62.41901
25	25	2	2	Bias	0.0488	0.02274	0.04821	0.04693
				MSE	0.00422	0.00135	0.00365	0.00267
				RE		68.0094787	13.50711	36.72986
50	25			Bias	0.0543	0.02012	0.02341	0.02204
				MSE	0.00424	0.00127	0.00268	0.00178
				RE		70.04716	36.79245	58.01887
10	50			Bias	0.07654	0.01218	0.02656	0.01496
				MSE	0.0015	0.00049	0.0011	0.0005
				RE		67.33333	26.66667	66.66667
25	50	2	3.5	Bias	0.09305	0.01339	0.0889	0.07917
				MSE	0.00738	0.00142	0.00725	0.00683
				RE		80.75880	1.76151	7.45257
50	50			Bias	0.09349	0.07497	0.08526	0.0813
				MSE	0.00815	0.00264	0.00758	0.00619
				RE		67.60736	6.99386	24.04908
10	10			Bias	0.09425	0.04107	0.06125	0.04187
				MSE	0.00872	0.00332	0.00392	0.0034
				RE		61.9266	55.04587	61.00917
25	10	3.5	0.5	Bias	0.06694	0.01247	0.06641	0.02858
				MSE	0.00474	0.00176	0.00456	0.00183
				RE		62.86913	3.79746	61.39241
50	10			Bias	0.05612	0.0122	0.05507	0.01545
				MSE	0.00373	0.00256	0.00368	0.00278
				RE		31.367292	1.34048	25.46917
10	25			Bias	0.05431	0.03443	0.04227	0.03449
				MSE	0.00767	0.00569	0.00621	0.00576
		3.5	2	RE		25.81486	19.0352	24.90222
25	25			Bias	0.013958	0.01142	0.01320	0.01151
				MSE	0.002163	0.0009	0.00167	0.00105
				RE		58.39112	22.79242	51.45631
50	25			Bias	0.08737	0.01456	0.03314	0.01478
				MSE	0.004929	0.00167	0.00319	0.00169
				RE		66.11888	35.28099	65.71313
10	50			Bias	0.06102	0.02032	0.02465	0.02313
				MSE	0.00871	0.00216	0.00835	0.00273
				RE		75.20091	4.13318	68.65672
25	50	3.5	3.5	Bias	0.0543	0.01212	0.01614	0.01239
				MSE	0.00387	0.00159	0.00236	0.00163
				RE		58.91472	39.01809	57.88114
50	50			Bias	0.0717	0.02231	0.06716	0.0268
				MSE	0.00308	0.0003	0.00212	0.00058
				RE		90.2597403	31.16883	81.16883

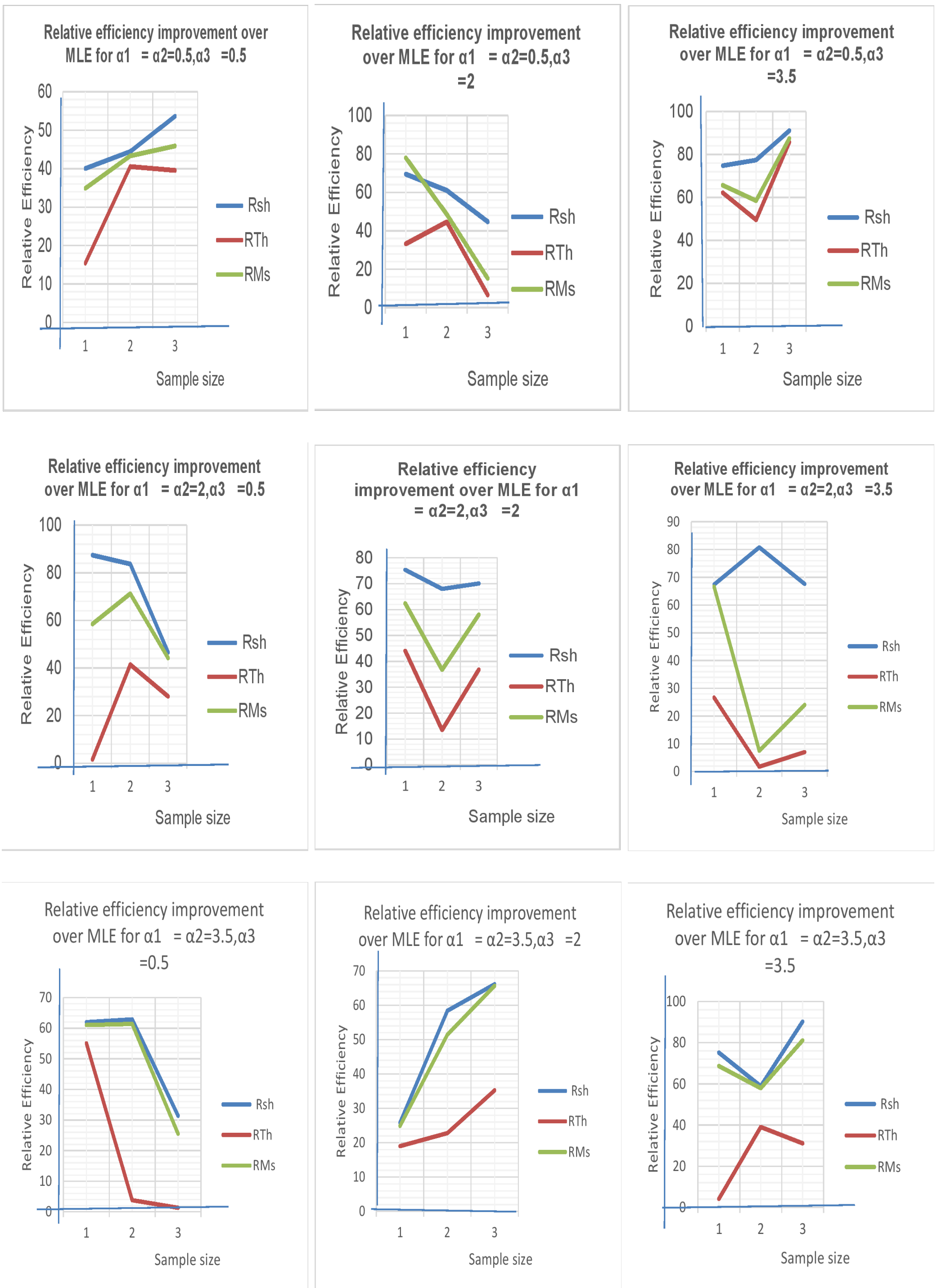

Fig. 3. Relative efficiency improvement Over MLE with complete sample using quasi Liklihood Estimation

Table5: Bias, MSE and Relative Efficiency of the estimates of Reliability functions under censored sample (30% censoring) using Quasi Likelihood Estimation.

n1=n2	n3	$\alpha_1 = \alpha_2$	α_3		RMLE	Rsh	RTh	RMs
				Bias	0.0195	0.01433	0.07033	0.018275
10	10			MSE	0.00173	0.00114	0.00169	0.00158
				RE		34.10405	2.312139	8.67052
				Bias	0.01829	0.0102	0.020104	0.013016
25	10	0.5	0.5	MSE	0.00152	0.00062	0.0009	0.000843
				RE		58.94737	40.78947	44.53947
				Bias	0.01437	0.02045	0.027	0.02047
50	10			MSE	0.00553	0.00101	0.0038	0.00109
				RE		81.73599	31.28391	80.28933
				Bias	0.04865	0.06245	0.08585	0.0826
10	25			MSE	0.00588	0.00143	0.00356	0.001639
				RE		75.68027	39.45578	72.12585
				Bias	0.0487	0.01149	0.08356	0.058
25	25	0.5	2	MSE	0.00498	0.00145	0.00477	0.00395
				RE		70.88353	4.216867	20.68273
				Bias	0.0556	0.012039	0.05738	0.01471
50	25			MSE	0.00562	0.00158	0.0038	0.00231
				RE		71.88612	32.38434	58.8968
				Bias	0.02005	0.0208	0.0473	0.0323
10	50			MSE	0.00488	0.00053	0.00123	0.00113
				RE		89.13934	74.79508	76.84426
				Bias	0.0202	0.02477	0.027	0.02047
25	50	0.5	3.5	MSE	0.00512	0.00109	0.00414	0.00225
				RE		78.71094	19.14063	56.05469
				Bias	0.02993	0.01632	0.03127	0.02954
50	50			MSE	0.00415	0.00111	0.00125	0.00123
				RE		73.25301	69.87952	70.36145
				Bias	0.08695	0.01198	0.04748	0.02108
10	10			MSE	0.0215	0.00101	0.00184	0.00121
		2	0.5	RE		95.30233	91.44186	94.37209
				Bias	0.1136	0.01061	0.017821	0.012828
25	10			MSE	0.01396	0.00114	0.00222	0.0017
				RE		91.83381	84.09742	87.82235

				Bias	0.08572	0.012935	0.073037	0.0529
50	10			MSE	0.0075	0.00399	0.00731	0.00627
				RE		46.8	2.533333	16.4
				Bias	0.02349	0.02238	0.03123	0.02348
10	25			MSE	0.00645	0.00357	0.00468	0.00449
				RE		44.65116	27.44186	30.3876
				Bias	0.04927	0.02269	0.04845	0.03935
25	25	2	2	MSE	0.00378	0.00135	0.00253	0.00221
				RE		64.28571	33.06878	41.53439
				Bias	0.05484	0.02204	0.0233	0.02209
50	25			MSE	0.00423	0.00113	0.002204	0.00147
				RE		73.28605	47.89598	65.24823
				Bias	0.00849	0.01106	0.02499	0.0114
10	50			MSE	0.00273	0.00145	0.00163	0.00156
				RE		46.88645	40.29304	42.85714
				Bias	0.07724	0.0339	0.06533	0.0652
25	50	2	3.5	MSE	0.0062	0.00372	0.00522	0.00475
				RE		40	15.80645	23.3871
				Bias	0.07525	0.03108	0.08647	0.08459
50	50			MSE	0.00811	0.00158	0.00754	0.0065
				RE		80.51788	7.02836	19.85203
				Bias	0.03866	0.012706	0.050516	0.04036
10	10			MSE	0.00498	0.00127	0.00331	0.00209
				RE		74.49799	33.53414	58.03213
				Bias	0.07141	0.024	0.09245	0.02858
25	10	3.5	0.5	MSE	0.00637	0.00316	0.0055	0.00456
				RE		50.39246	13.65777	28.41444
				Bias	0.04824	0.015309	0.04924	0.03101
50	10			MSE	0.00326	0.00215	0.00281	0.00229
				RE		34.04908	13.80368	29.7546
				Bias	0.03869	0.0234	0.03425	0.02626
10	25			MSE	0.00779	0.00558	0.00722	0.00648
		3.5	2	RE		28.3697	7.317073	16.81643
				Bias	0.00101	0.02974	0.06911	0.06182
25	25			MSE	0.0052	0.00128	0.00382	0.00154
				RE		75.38462	26.53846	70.38462
				Bias	0.01464	0.01377	0.03351	0.0175
50	25			MSE	0.00384	0.00129	0.00216	0.00155
				RE		66.40625	43.75	59.63542
				Bias	0.02135	0.01184	0.01074	0.01899
10	50			MSE	0.0022	0.00078	0.00197	0.00126
				RE		64.54545	10.45455	42.72727
				Bias	0.0115	0.01212	0.02891	0.01276
25	50	3.5	3.5	MSE	0.00228	0.00117	0.00154	0.00123
				RE		48.68421	32.45614	46.05263
				Bias	0.00351	0.01009	0.0216	0.017073
50	50			MSE	0.00346	0.000212	0.000578	0.00028
				RE		93.87283	83.2948	91.90751

Table6 : Bias, MSE and Relative Efficiency of the estimates of Reliability functions under censored sample (50% censoring) using Quasi Likelihood Estimation.

$n_1=n_2$	n_3	$\alpha_1 = \alpha_2$	α_3		RMLE	Rsh	RTh	RMs
				Bias	0.02334	0.01433	0.07033	0.018275
10	10			MSE	0.00174	0.00114	0.00169	0.00158
				RE		34.48276	2.873563	9.195402
				Bias	0.01782	0.0102	0.020104	0.013016
25	10	0.5	0.5	MSE	0.00146	0.000624	0.0009	0.000843
				RE		57.26027	38.35616	42.26027
				Bias	0.01368	0.02045	0.027	0.02047
50	10			MSE	0.00435	0.00101	0.0038	0.00109
				RE		76.78161	12.64368	74.94253
				Bias	0.05312	0.06245	0.08585	0.0826
10	25			MSE	0.00536	0.00143	0.00356	0.001639
				RE		73.3209	33.58209	69.42164
				Bias	0.04922	0.01149	0.08356	0.058
25	25	0.5	2	MSE	0.00594	0.00145	0.00477	0.00395
				RE		75.58923	19.69697	33.50168
				Bias	0.05607	0.012039	0.05738	0.01471
50	25			MSE	0.00665	0.00158	0.0038	0.00231
				RE		76.2406	42.85714	65.26316
				Bias	0.02511	0.0208	0.0473	0.0323
10	50			MSE	0.0075	0.00053	0.00123	0.00113
				RE		92.93333	83.6	84.93333
				Bias	0.02041	0.02477	0.027	0.02047
25	50	0.5	3.5	MSE	0.0049	0.00109	0.00414	0.00225
				RE		77.7551	15.5102	54.08163
				Bias	0.03031	0.01632	0.03127	0.02954
50	50			MSE	0.00517	0.00111	0.00125	0.00123
				RE		78.52998	75.82205	76.2089
				Bias	0.08987	0.01198	0.04748	0.02108
10	10			MSE	0.00506	0.00101	0.00184	0.00121
		2	0.5	RE		80.03953	63.63636	76.08696
				Bias	0.12218	0.01061	0.017821	0.012828
25	10			MSE	0.01583	0.00114	0.00222	0.0017
				RE		92.79848	85.97599	89.2609

				Bias	0.09082	0.012935	0.073037	0.0529
50	10			MSE	0.00846	0.00399	0.00731	0.00627
				RE		52.83688	13.59338	25.88652
				Bias	0.01579	0.02238	0.03123	0.02348
10	25			MSE	0.00487	0.00357	0.00468	0.00449
				RE		26.69405	3.901437	7.802875
				Bias	0.04691	0.02269	0.04845	0.03935
25	25	2	2	MSE	0.0035	0.00135	0.00253	0.00221
				RE		61.42857	27.71429	36.85714
				Bias	0.05359	0.02204	0.0233	0.02209
50	25			MSE	0.00393	0.00113	0.002204	0.00147
				RE		71.24682	43.91858	62.59542
				Bias	0.00303	0.01106	0.02499	0.0114
10	50			MSE	0.00227	0.00145	0.00163	0.00156
				RE		36.12335	28.19383	31.27753
				Bias	0.07594	0.0339	0.06533	0.0652
25	50	2	3.5	MSE	0.00588	0.00372	0.00522	0.00475
				RE		36.73469	11.22449	19.21769
				Bias	0.07217	0.03108	0.08647	0.08459
50	50			MSE	0.00854	0.00158	0.00754	0.0065
				RE		81.49883	11.7096	23.88759
				Bias	0.04945	0.012706	0.050516	0.04036
10	10			MSE	0.01001	0.00127	0.00331	0.00209
				RE		87.31269	66.93307	79.12088
				Bias	0.06274	0.024	0.09245	0.02858
25	10	3.5	0.5	MSE	0.00584	0.00316	0.0055	0.00456
				RE		45.89041	5.821918	21.91781
				Bias	0.04651	0.015309	0.04924	0.03101
50	10			MSE	0.00319	0.00215	0.00281	0.00229
				RE		32.60188	11.91223	28.21317
				Bias	0.0267	0.0234	0.03425	0.02626
10	25			MSE	0.00812	0.00558	0.00722	0.00648
				RE		31.28079	11.08374	20.19704
		3.5	2	Bias	0.088	0.02974	0.06911	0.06182
25	25			MSE	0.0051	0.00128	0.00382	0.00154
				RE		74.90196	25.09804	69.80392
				Bias	0.01529	0.01377	0.03351	0.0175
50	25			MSE	0.00384	0.00129	0.00216	0.00155
				RE		66.40625	43.75	59.63542
				Bias	0.01637	0.01184	0.01074	0.01899
10	50			MSE	0.00212	0.00078	0.00197	0.00126
				RE		63.20755	7.075472	40.56604
				Bias	0.01661	0.01212	0.02891	0.01276
25	50	3.5	3.5	MSE	0.00197	0.00117	0.00154	0.00123
				RE		40.60914	21.82741	37.56345
				Bias	0.00172	0.01009	0.0216	0.017073
50	50			MSE	0.00093	0.000212	0.000578	0.00028
				RE		77.2043	37.84946	69.89247

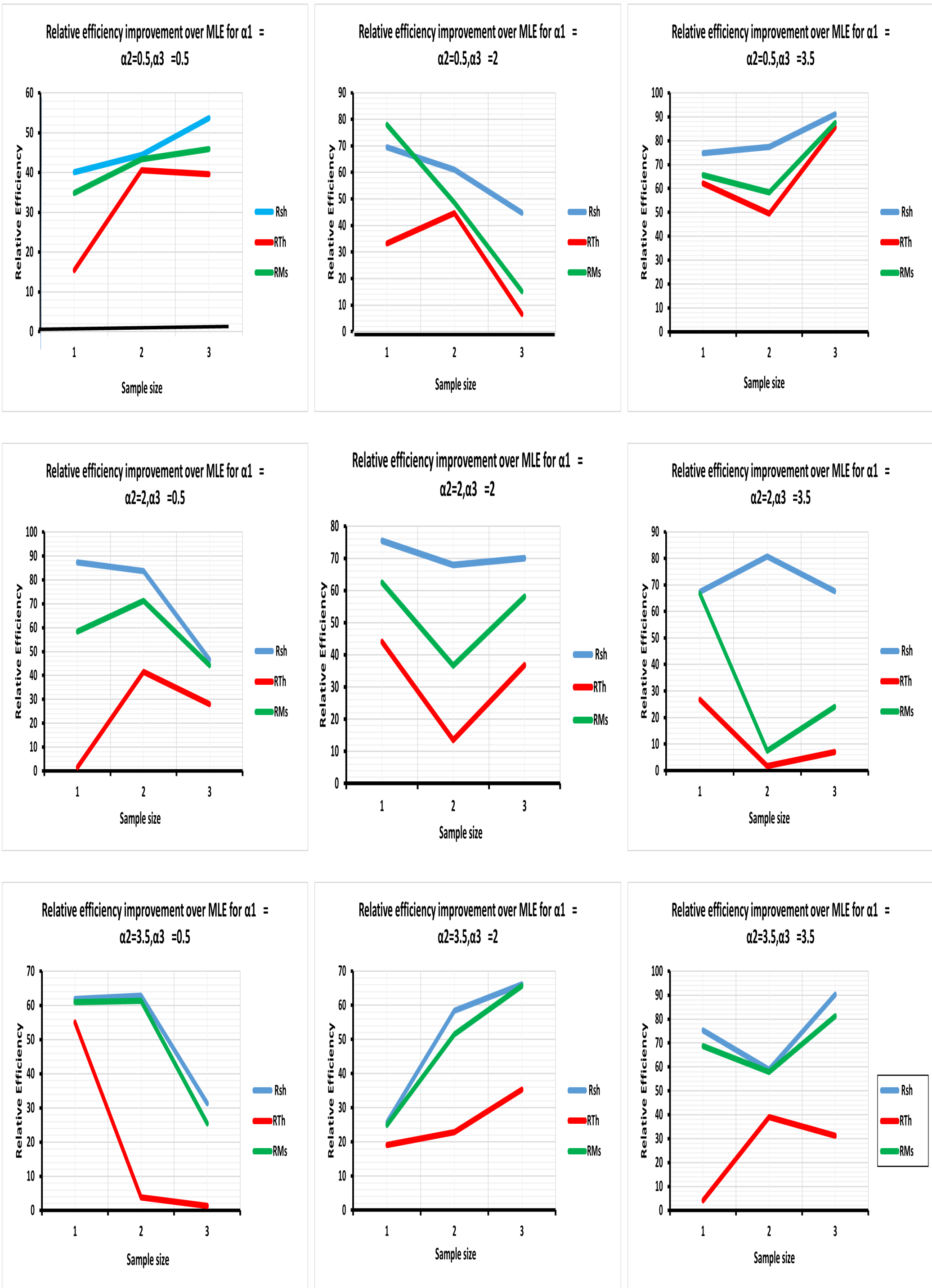

Fig. 4. Relative efficiency improvement Over MLE with censored sample using quasi Liklihood Estimation

Table 7: Confidence Interval for R

$n_1 = n_2$	n_3	$\alpha_1 = \alpha_2$	α_3	CI based on Maximum Likelihood		CI based on Quasi Likelihood	
				Complete Sample	Censored Sample	Complete Sample	Censored Sample
10	10	0.5	0.5	(0.28855,0.31636)	(0.15350,0.16600)	(0.09297,0.22206)	(0.10668,0.21231)
	25	0.5	2	(0.07304,0.07904)	(0.03121,0.03312)	(0.01313,0.13004))	(0.00627,0.13539)
	50	0.5	3.5	(0.23386,0.25177)	(0.01306,0.02264)	(0.00535,0.05423))	(0.01295,0.07302)
25	10	0.5	0.5	(0.21772,0.25318)	(0.11332,0.16304)	(0.08964,0.22769)	(0.10724,0.21307)
	25	0.5	2	(0.05207,0.05980)	(0.01741,0.03321)	(0.01625,0.13186)	(0.01450,0.13917)
	50	0.5	3.5	(0.04857,0.05936)	(0.01305,0.02863)	(0.00787,0.05611)	(0.00689,0.06164))
50	10	0.5	0.5	(0.21337,0.22904)	(0.04533,0.16615)	(0.15286,0.17211)	(0.09498,0.22524)
	25	0.5	2	(0.05090,0.05340)	(0.01130,0.03327)	(0.01110,0.13303)	(0.01115,0.13520)
	50	0.5	3.5	(0.05319,0.06132)	(0.14523,0.30152)	(0.01692,0.07439)	(0.01359,0.05937)
10	10	2	0.5	(0.17594,0.55593)	(0.29699,0.33841)	(0.28300,0.34679)	(0.23508,0.35615)
	25	2	2	(0.08069,0.25904)	(0.07523,0.16638)	(0.15948.0.20268)	(0.05799.0.29771)
	50	2	3.5	(0.01291,0.19987)	(0.04228,0.09435)	(0.09419,0.11192)	(0.03385,0.18506)
25	10	2	0.5	(0.25560,0.49728)	(0.34512,0.37807)	(0.24960,0.35355)	(0.27487,0.30795)
	25	2	2	(0.11002,0.23800)	(0.16217,0.21468)	(0.16406,0.21748)	(0.10514.0.26754)
	50	2	3.5	(0.05560,0.15998)	(0.08384,0.17286)	(0.08859,0.16556)	(0.12482,0.14317)
50	10	2	0.5	(0.22044,0.49194)	(0.27315,0.34092)	(0.27066,0.35533)	(0.29460,0.33182)
	25	2	2	(0.18821,0.58189)	(0.10555,0.16458)	(0.16412,0.22350)	(0.10510,0.19858)
	50	2	3.5	(0.04538,0.14877)	(0.09538,0.12355)	(0.09280.0.16555)	(0.12089,0.12784)
10	10	3.5	0.5	(0.20326,0.54997)	(0.28368,0.51137)	(0.32861,0.44617)	(0.20993,0.55621)
	25	3.5	2	(0.16684,0.28309)	(0.14396,0.32620)	(0.11589,0.41648)	(0.08899,0.43220)
	50	3.5	3.5	(0.13660,0.15472)	(0.04230,0.32455)	(0.07480,0.28166)	(0.10906,0.23500)
25	10	3.5	0.5	(0.34019,0.47313)	(0.28030,0.60112)	(0.37280,0.37860)	(0.35622,0.36916)
	25	3.5	2	(0.14997,0.34598)	(0.16010,0.25589)	(0.20404,0.29204)	(0.21082,0.27793)
	50	3.5	3.5	(0.24106,0.30581)	(0.05483,0.36377)	(0.09132,0.22961)	(0.33681,0.44883)
50	10	3.5	0.5	(0.31777,0.48834)	(0.26758,0.52560)	(0.36103,0.39950)	(0.35193,0.44120)
	25	3.5	2	(0.01268,0.11261)	(0.31249,0.83280)	(0.17222,0.33750)	(0.15575,0.31094)
	50	3.5	3.5	(0.25716,0.33736)	(0.01154,0.26594)	(0.13524,0.20525)	(0.12838,0.18257)

8 CONCLUSION

From the numerical study conducted so for we can conclude that

- When sample size increases bias and mean square error decreases.
- The relative efficiency improvement over MLE of the $\hat{R}_{sh}$ greater than that of $\hat{R}_{Th}$ and $\hat{R}_{Ms}$. So $\hat{R}_{sh}$ is performs better than $\hat{R}_{Th}$ and $\hat{R}_{Ms}$.
- In Maximum Likelihood Estimation when sample sizes is large the width of confidence interval of $\hat{R}_{mle}$ is less than that of $\hat{R}_{mlec}$.
- In Quasi Likelihood Estimation when sample sizes is large the width of confidence interval $\hat{R}_{qmle}$ is less than that of $\hat{R}_{qmlec}$.
- Relative efficiency improvement over MLE is higher in the case of censored sample

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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