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Limit Cycle for the RC Circuit with Square-Wave-Type Voltage

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Abstract

The goal of this paper is to prove, by means of calculations the existence of the ω-limit cycle of a simple circuit where a resistor, a capacitor and a square-wave-type voltage source are in series.

Keywords

DC RC Circuit, One-Sided Limits, Geometric Series

1. Introduction

An RC circuit connects resistors and capacitors and when contains only sources of constants values is a DC (Direct current) RC circuit. The DC RC circuit is widely treated in undergraduate Physics university texts [1] and in Physics laboratories, including virtual laboratories. The theory of the simple DC RC circuit that contains in series a source of constant value, a resistor and a capacitor will be one of the key tools of this present work. RC circuit is also applied as an engineering tool. In alternating current in which the sources are sinusoidal in time, for example, the AC RC circuit where the source, resistor and capacitor are in series is used as filter in the frequency or time domain. RC circuit may also be used as a tool to study the percolation phenomena [2]. In addition, the RC circuit is a model for various phenomena. In Neuroscience, for example, the neural cell membrane behaves like an RC circuit [3]. In Physics and Chemistry, a liquid crystal cell can be modeled as a variable capacitor and studied in an RC circuit [4]. In quantum mechanics, we find the quantum analogs of the RC circuit, one based on charge and the other on spin [5]. Here it will consider a typical physics' problem about RC circuit (**Figure 1**) to transform it to a mathematical study applying the DC RC circuit. The advantage of the physical point of view is that the prediction can be easily probed experimentally.

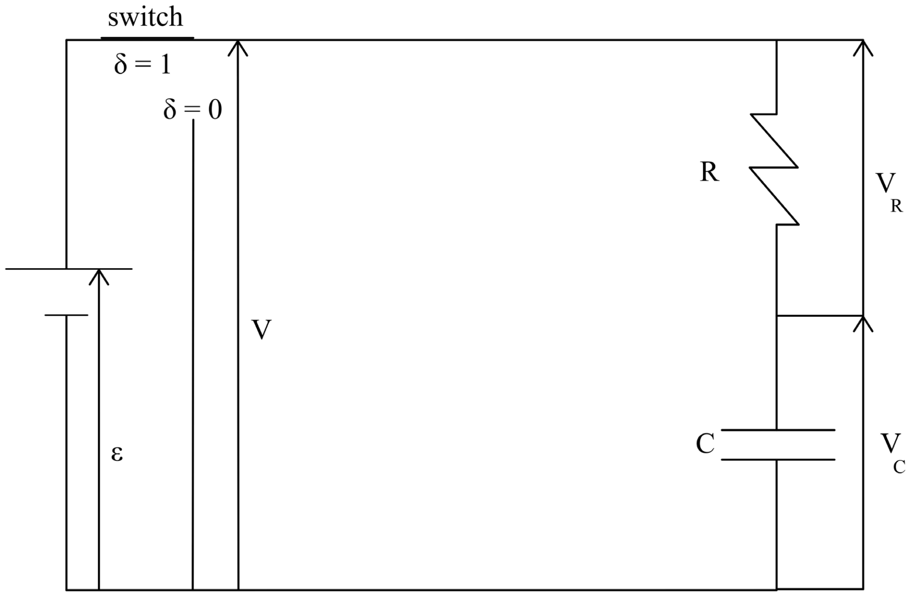

Figure 1. GE's circuit start at time=0, switch's on in $\delta = 1$ and the capacitor contains a charge Q_0 on one plate.

In Physics, for mechanical systems, the phase space is a plot of position and momentum variables as function of time. Given an initial condition in the phase space there is a trajectory in time. For a system may exist an especial trajectory called stable or attractive limit cycle ω because all the neighboring trajectories approach to it when the time is very long. Finding limit cycles, in general, is a very difficult problem but important in dynamical system, differential equations, control theory and its applications as models to study the behavior of many real-world oscillatory systems in Aerodynamics, the Hodgkin-Huxley model for action potential in neurons and many others. This work treats about the study in the phase space defined by the electrical charge, current and time that corresponds to the RC circuit with square-wave-type voltage (**Figure 2**).

In other words, given an initial condition on the charge stored in the capacitor Q_0 of the GE's RC circuit, what would be seen if the subsequent values of the charge Q and the electric current I are plotted over a very long time at move the switch alternately and indefinitely between $\delta = 1$ and $\delta = 0$? The GE is carried out to answer this question. However, the GE will be develop little different to the usual physical way but parallel to it—mathematical treatment—via the introduction of the natural quantities source voltage ε, resistor's resistance R together the capacitor's capacitance C that are more adequate to the problem at hands. Theses quantities ε, R, C allow us to define

$$\tau = RC \tag{1}$$

as time's unit and the next dimensionless quantities

$$t = \frac{time}{\tau}, \quad q = \frac{Q}{\varepsilon C}, \quad \dot{q} = \frac{I}{\varepsilon / R}, \tag{2}$$

that corresponds—respectively—to the time, the charge and the current. Thus

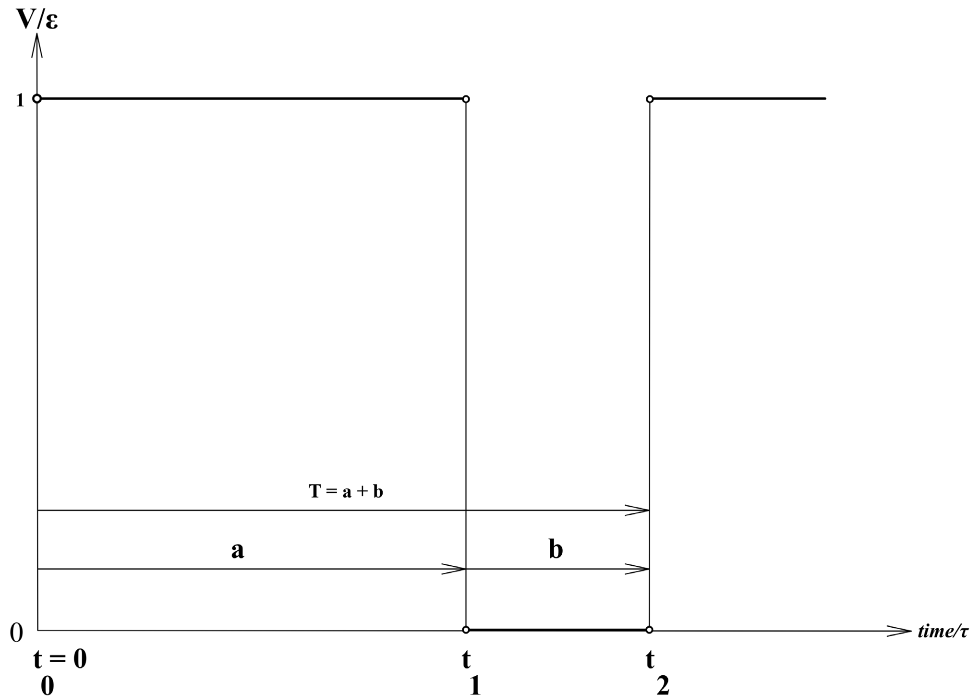

Figure 2. The square-wave-type voltage.

this way any trajectory γ (orbit) will be the set of vector's state

$$\boldsymbol{r} = \boldsymbol{r}(t) = \left(q(t), \dot{q}(t)\right) \tag{3}$$

determined by the GE's RC circuit in the interval time $0 \le t \le \infty$, $t \ne t_{n=0,1,2,\cdots}$,

$$t_n = \begin{Bmatrix} t_{2m} = mT \\ t_{2m+1} = mT + a \end{Bmatrix}, \quad T = a + b, \quad m = 0,1,2,\cdots \tag{4}$$

in the phase space for a given initial condition in $t_0 = 0$

$$\boldsymbol{r}_0 = \left(q_0, \dot{q}_0 = -q_0 + 1\right). \tag{5}$$

But, what's meaning GE? At is core, the GE will be a theoretical process to calculate $q = q(t)$, $\dot{q} = \dot{q}(t)$ for the circuit of **Figure 1** under the initial condition q_0. Between the GE's results will need the others key tools that we will need to manipulate, the one-sided limits and the geometric series. They are old in mathematics and its development, since they involve the relevant concepts of infinity and limit [6]. Numerous examples and applications involving geometric series can be found in physics, engineering, biology, economics, computer science, queueing theory, medicine [7] and finance.

More precisely, the steps of the GE will be as follow: introduction, toy model and results.

- The GE's introduction refers to a shortly review of the GE's RC circuit using the theory of DC RC circuit and the Landau's dimensionless criterion [8] to define more dimensionless quantities and the linear differential equations or state's equations

$$\dot{q} = -q + \delta. \tag{6}$$

State's equations define how the system's state changes over time t into every open intervals $t_{n-1} < t < t_n = t_{n-1} + \theta$ of length a or b denoted as

$$(\theta) = (t_{n-1}, t_n = t_{n-1} + \theta), \tag{7}$$

where $\theta = a$ if $n = 1, 3, 5, \cdots$ or $\theta = b$ if $n = 2, 4, 6, \cdots$, and the piecewise constant function

$$\delta = \delta(t) = \begin{cases} 1 & t \in (a) \\ 0 & t \in (b) \end{cases}. \tag{8}$$

Then with the auxiliary variable $x = t - t_{n-1}$, the solutions can be written as

$$q(t) = c\mathrm{e}^{-x} + \delta(1 - \mathrm{e}^{-x}) = \delta + (c - \delta)\mathrm{e}^{-x}. \tag{9}$$

- Toy model. Up to this point it is possible to find the charge and current within each open intervals (a) or (b), but this is where the real problem begins because something else is missing. In the previous step it was not contemplate the knowing about q and $\dot{q}$ in every t_n. It's necessary a special attention when $t = t_n$ because there may exist discontinuity. Taken in account the discontinuity's analyses through the famous one-sided limits are defined:

$$q_{in\ \theta}^{n-1} = \lim_{t \to t_{n-1}^{+}} q(t) = \lim_{x \to 0^{+}} q = q(t_{n-1}+) \tag{10}$$

in terms of the right-handed limit, and

$$q_{out\ \theta'}^{n} = \lim_{t \to t_n^{-}} q(t) = \lim_{x \to \theta'^{-}} q = q(t_n -) \tag{11}$$

using the left-handed limit.

Physically, it is to be expected that the amount of electric charge in the circuit is conserved, that is, the $q(t)$ must be continuous all time. In particular when $t = t_n$

$$q_{out\ \theta' \neq \theta}^{n} = q(t_n) = q_n = q_{in\ \theta}^{n}. \tag{12}$$

For the current are defined in a similar way,

$$\tilde{q}_{in\ \theta}^{n-1} = \lim_{t \to t_{n-1}^{+}} \dot{q}(t) = \lim_{x \to 0^{+}} \dot{q} = \dot{q}(t_{n-1}+) \tag{13}$$

and

$$\tilde{q}_{out\ \theta'}^{n} = \lim_{t \to t_n^{-}} \dot{q}(t) = \lim_{x \to \theta'^{-}} \dot{q} = \dot{q}(t_n -). \tag{14}$$

However, the fact that the switch change implies that the current $\dot{q}(t)$ is discontinuous at every time t_n,

$$\tilde{q}_{out\ \theta' \neq \theta}^{n} \neq \tilde{q}_{in\ \theta}^{n}. \tag{15}$$

The quantities

$$0 < A := \mathrm{e}^{-a} < 1, \tag{16}$$

$$0 < B := \mathrm{e}^{-b} < 1 \quad \text{and} \tag{17}$$

$$0 < F := 1 - A < 1 \tag{18}$$

will be frequently used. Summarizing, for example, for a time's lap of one period

$a+b$, the calculus of the charge are

$$\begin{aligned} &u=q_{2m}=q_{in\,a}^{2m}\Rightarrow uA+F=v=q_{out\,a}^{2m+1}=q_{2m+1}=q_{in\,b}^{2m+1} \\ &\Rightarrow vB=w=q_{out\,b}^{2(m+1)}=q_{2(m+1)}=q_{in\,a}^{2(m+1)}\cdots \end{aligned} \tag{19}$$

and for the current are

$$\begin{aligned} &\tilde{q}_{in\,a}^{2m}=-u+1\Rightarrow \tilde{q}_{out\,a}^{2m+1}=-v+1\neq \tilde{q}_{in\,b}^{2m+1}=-v \\ &\Rightarrow \tilde{q}_{out\,b}^{2(m+1)}=-w\neq \tilde{q}_{in\,a}^{2(m+1)}=-w+1\cdots \end{aligned} \tag{20}$$

So, it is invented a toy model [9] (**Figure 3**) to resume and work easier between every period of time $T=a+b$. Too, this toy model can help build physical intuition. At this context, a toy model was created—as an electrical analogy inspired in the particles' collision—to concentrate the attention in q_n, $\tilde{q}_{out\,\theta'\neq\theta}^{n}$, $\tilde{q}_{in\,\theta}^{n}$. This toy model may allow analyzing and predicting the system's future states.

- Results. Now it is possible to change on to a second stage using toy model. Here is putting in the spotlight to the quantity q_{2m+1} because if it is known its expression in terms of m it will know the rest of the others quantities required in the phase space. To motivate a probably expression for q_{2m+1} it is obtained iteratively some test expression of q_{2m+1} using toy model. Working until to reduce it to a desire form it will deduce as probably expression

$$q_{2m+1}=q_1r^m+s_m \tag{21}$$

In the last equation is identified

$$q_1=q_0A+F \tag{22}$$

together

$$s_m=\begin{cases}0 & m=0\\ F+Fr+\cdots+Fr^{m-1} & m\geq 1\end{cases} \tag{23}$$

as the sum of the first m term of a geometric series, up to and including the Fr^{m-1} term, which is giving by the formula

$$s_m=\frac{F}{1-r}\left(1-r^m\right), m\geq 0 \tag{24}$$

where

$$0<r:=BA=\mathrm{e}^{-T}<1 \tag{25}$$

is the positive common ratio less than unity. As the number m approaches to infinity the series $\{s_m;m=0,1,2,\cdots\}$ converges to

$$s=\frac{F}{1-r}. \tag{26}$$

The partial sum s_m can be written as $s_m=s\left(1-r^m\right)$, therefore the GE's concludes as its main result with the equation

$$q_{2m+1}=pr^m+s \tag{27}$$

where

$$p = (q_0 - sB)A. \tag{28}$$

By strong induction on m it will probe that the test expression $q_{2m+1} = pr^m + s$ is right. All other information on the vector's state $(q(t), \dot{q}(t))$ can be derived from the above equations. For example, the ω-limit cycle (see **Figure 4**) is giving by

$$q_n = \begin{Bmatrix} q_{2m} = sB := q_b^\omega = \dfrac{e^{-b} - e^{-T}}{1 - e^{-T}} \\ q_{2m+1} = s := q_a^\omega = \dfrac{1 - e^{-a}}{1 - e^{-T}} \end{Bmatrix} \quad m = 0,1,2,\cdots \tag{29}$$
$$t_n < t = t_n + x < t_{n+1} = t_n + \theta \quad q = q_n e^{-x} + \delta(1 - e^{-x}) \quad \boldsymbol{r} = (1,-1)q + (0,1)\delta$$

Therefore when $q_0 = sB, p = 0$ occurs the ω-limit cycle. Other trajectories in the phase space can be visualized in **Figure 4(a)** via the information in **Figure 4(b)**. Finally, it is possible to propose as answer that the total response of the GE's RC circuit in the phase space on the time $0 \le t \le \infty$, $t \ne t_{n=0,1,2,\cdots}$ for a given initial condition

$$\boldsymbol{r}_0 = (q_0, -q_0 + 1) \tag{30}$$

in $t_0 = 0$ will be the trajectory γ for which $\boldsymbol{r} = \boldsymbol{r}(t) = (q(t), \dot{q}(t))$ will be state's vector such that

$$q_n = \begin{Bmatrix} q_0 & . \\ q_{2m+1} = pr^m + q_a^\omega & m = 0,1,2,\cdots \\ q_{2(m+1)} = pBr^m + q_b^\omega & . \end{Bmatrix} \quad p = (q_0 - q_b^\omega)A \tag{31}$$
$$t_n < t = t_n + x < t_{n+1} = t_n + \theta \quad q = q_n e^{-x} + \delta(1 - e^{-x}) \quad \boldsymbol{r} = (1,-1)q + (0,1)\delta$$

2. The Gedanken Experiment

The proof of what was previously predicted is based in the GE. **Figure 1** shows the GE's RC circuit that employs

- the battery creates a constant potential difference $\varepsilon \ne 0$;
- the two-positions switch is initially at the position $\delta = 1$;
- if capacitor's capacitance is C, Q symbolizes the maximum charge that can be stored when the applied voltage is V_C across its plates then the capacitor's capacitance is

$$C = \frac{Q}{V_C}. \tag{32}$$

The Q_0 is the initial stored charge in the capacitor; and

- if the I is the rate at which the electrical charge flows through an area of the ohmic resistor of resistance R when the potential drop across this device is V_R then the magnitude of the typical material's resistance (Ohm's law) is

$$R = \frac{V_R}{I}. \tag{33}$$

The switch start when the $time = 0$ in the position $\delta = 1$, where remains there during a time $a\tau$ changes instantaneously to $\delta = 0$ where it remains a time $b\tau$ and so on alternately between the two positions and indefinitely. Consequently, during the GE, the RC circuit is subjected to a square-wave-type voltage V source, as illustrated in **Figure 2** an RC circuit with a two-pole switch. Some dimensionless quantities will be introduced soon. It starts with the input voltage source or input signal which expression is giving by

$$\frac{V}{\varepsilon} = \delta . \tag{34}$$

The voltage across the capacitor is

$$\frac{V_C}{\varepsilon} = \frac{Q}{\varepsilon C} = q \tag{35}$$

and the potential drop across the resistor is

$$\frac{V_R}{\varepsilon} = \frac{I}{\varepsilon/R} = \frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q} . \tag{36}$$

According to the conservation of energy or Kirchhoff's loop rule

$$V = V_C + V_R \tag{37}$$

but using the above equations it can be written that

$$\dot{q} = -q + \delta \tag{38}$$

in every open interval (a) or (b). The solutions $q(t)$ of the linear differential equation or the total response of the DC RC circuit

$$q = c\mathrm{e}^{-x} + \delta\left(1 - \mathrm{e}^{-x}\right). \tag{39}$$

2.1. Toy Model

As previously written, this section is dedicated to the analysis of possible discontinuities and culminates with a mathematical model called "toy model" that will be used later. Substituting the previous expression of the charge $q(x + t_n)$ in the one-sided limits, it was find that

$$\begin{aligned} q_{in\,\theta}^{n-1} &= \lim_{t \to t_{n-1}^+} q = \lim_{x \to 0^+} q = q_{n-1} = c = \begin{cases} q_{in\,a}^{n-1} \\ q_{in\,b}^{n-1} \end{cases}, \\ q_{out\,\theta}^{n} &= \lim_{t \to t_n^-} q = \lim_{x \to \theta^-} q = \begin{cases} q_{out\,a}^{n} = cA + F \\ q_{out\,b}^{n} = cB \end{cases} \end{aligned} \tag{40}$$

where the quantities

$$0 < A = \mathrm{e}^{-a}, F = 1 - \mathrm{e}^{-a}, B = \mathrm{e}^{-b} < 1 \tag{41}$$

start to appear. If the charge in the circuit is conserved at any instant, particularly at $t = t_n$ must check there

$$\lim_{t \to t_n^-} q = q_{out\,\theta' \neq \theta}^{n} = q(t_n) = q_n = q_{in\,\theta}^{n} = \lim_{t \to t_n^+} q. \tag{42}$$

The charge's conservation implies that the solution is continuous all the time, *i.e.*:

$$\lim_{\bar{t}\to t} q = q(t) \tag{43}$$

This means that for the charge

$$q(t) = c\mathrm{e}^{-x} + \delta\left(1-\mathrm{e}^{-x}\right) \quad 0 \le x = t - t_n \le \theta\,. \tag{44}$$

Let's take a look at the current introducing the results obtained before on the charge's continuity

$$\tilde{q}^{n}_{in\ \theta} = \lim_{t\to t_n^+} \dot{q} = \lim_{x\to 0^+} \dot{q} = -q^{n}_{in\ \theta} + \lim_{x\to 0^+} \delta = -q_n + \lim_{t\to t_n^+} \delta\,, \tag{45}$$

$$\tilde{q}^{n}_{out\ \theta'\neq\theta} = \lim_{t\to t_n^-} \dot{q} = \lim_{x\to \theta'^-} \dot{q} = -q^{n}_{out\ \theta'\neq\theta} + \lim_{t\to t_n^-} \delta = -q_n + \lim_{t\to t_n^-} \delta \tag{46}$$

These quantities show that $\dot{q}$ is not continuous, because

$$\lim_{t\to t_n^-} \delta = \delta\left(t_n -\right) \neq \lim_{t\to t_n^+} \delta = \delta\left(t_n +\right). \tag{47}$$

Quantities $\tilde{q}$ will be useful since they coincide with $\dot{q}$, except in every t_n. Then, all the above can be summarized in the following expression:

$$\begin{aligned} q_{2m} &= q_{2m-1}B \qquad \begin{cases} \tilde{q}^{2m}_{out\ b} = -q_{2m} \\ \tilde{q}^{2m}_{in\ a} = -q_{2m} + 1 \end{cases} \\ q_{2m+1} &= q_{2m}A + F \qquad \begin{cases} \tilde{q}^{2m+1}_{out\ a} = -q_{2m+1} + 1 \\ \tilde{q}^{2m+1}_{in\ b} = -q_{2m+1} \end{cases} \end{aligned} \tag{48}$$

As an alternative to build the physical intuition and to realize the numerous calculus that will be obtain in the GE's results it was develop the flowchart or toy model below at **Figure 3**.

2.2. Results

After several trials in the paper with pencil and eraser applying expressions and calculations with the toy model for various periods of time, seeking to find some probable expression that could be familiar and "easy" to manipulate in the future, it was considered that a possible one is the one given by the following calculations:

$$q\left(t_0 = 0\right) = q_0 = \frac{Q_0}{\varepsilon C}$$

$$q\left(t_1 = 0T + a\right) = q_1 = q_0 A + F$$

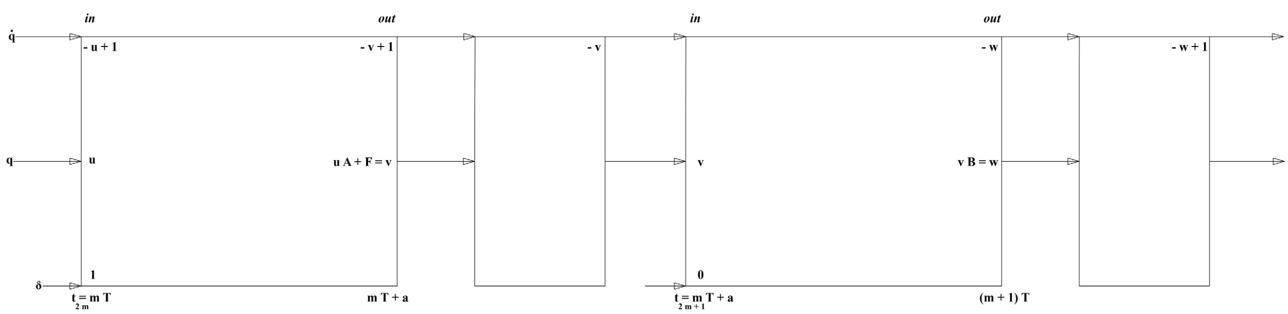

Figure 3. Iterative toy model over a period of time.

$$q(t_2 = 1T) = q_2 = q_1 B$$

$$q(t_3 = 1T + a) = q_3 = q_2 A + F = q_1 \cdot BA + F \cdot 1$$

$$q(t_4 = 2T) = q_4 = q_3 B = q_1 \cdot BA \cdot B + F \cdot B$$

$$q(t_5 = 2T + a) = q_5 = q_4 A + F = q_1 \cdot BA \cdot BA + F \cdot BA + F \cdot 1$$

$$q_6 = q_5 B = q_1 \cdot BA \cdot BA \cdot B + F \cdot BA \cdot B + F \cdot B$$

$$q_7 = q_1 \cdot BA \cdot BA \cdot BA + F \cdot BA \cdot BA + F \cdot BA + F \cdot 1$$

$$q_8 = q_7 B$$

$$q_9 = q_1 \cdot (BA)^4 + F \cdot (BA)^3 + F \cdot (BA)^2 + F \cdot (BA)^1 + F \cdot 1$$

$$q_{10} = q_9 B$$

$$q_{11} = q_1 \cdot (BA)^5 + F \cdot (BA)^4 + F \cdot (BA)^3 + F \cdot (BA)^2 + F \cdot (BA)^1 + F \cdot 1$$

$$q_{12} = q_{11} B$$

$$q_{13} = q_1 \cdot (BA)^5 + F \cdot 1 + F \cdot (BA)^1 + F \cdot (BA)^2 + F \cdot (BA)^3 + F \cdot (BA)^4$$

$$q_{2\cdot 7} = q_{2\cdot 7-1} B$$

$$q_{2.7+1} = q_1 \cdot (BA)^7 + F \cdot (BA)^0 + \cdots + F \cdot (BA)^{7-1}$$

$$q_{2\cdot 8} = q_{2\cdot 8-1} B$$

$$q_{2\cdot 8+1} = q_1 \cdot (BA)^8 + \sum_{j=0}^{8-1} F (BA)^j$$

They suggest that the desired expression could be of the form:

$$q_{2\cdot m+1} = q_1 \cdot (BA)^m + \sum_{j=0}^{m-1} F (BA)^j \tag{49}$$

while the expression for $q_{2\cdot(m+1)}$ it will be $q_{2\cdot m+1} \cdot B$, that is

$$q_{2\cdot(m+1)} = q_{2\cdot m+1} \cdot B. \tag{50}$$

In other words, the relation between $q_{2\cdot m+1}$ and $q_{2\cdot(m+1)}$ is known. Therefore it will be enough to find a formula for the cases in which $n = 1, 3, \cdots, 2m+1, \cdots$. Therefore, the above reasoning implies focusing on $q_{2\cdot m+1} = q_1 \cdot (BA)^m + \sum_{j=0}^{m-1} F (BA)^j$. This last formula suggests the definition of the following quantities

$$0 < r = BA = \mathrm{e}^{-T} < 1 \tag{51}$$

that will be important together the sum of the first m term of a geometric series, up to and including the Fr^{m-1} term

$$s_m = F + Fr + \cdots + Fr^{m-1} = \frac{F}{1-r}\left(1 - r^m\right). \tag{52}$$

Therefore $q_{2\cdot m+1} = q_1 \cdot (BA)^m + \sum_{j=0}^{m-1} F (BA)^j$ it may write as

$$q_{2m+1} = q_1 r^m + s_m.$$

But since the geometric series with ratio r is convergent to

$$s = \frac{F}{1-r}, \quad 0 < s < 1 \tag{53}$$

by substituting $\frac{F}{1-r}$ for s in their partial sum $\frac{F}{1-r}\left(1-r^m\right)$ and rearranging the formula $q_{2m+1} = q_1 r^m + s_m$, leads to:

$$q_{2m+1} = pr^m + s, \quad p = \left(q_0 - sB\right)A. \tag{54}$$

Is the formula $q_{2m+1} = pr^m + s$ correct? It will to show by induction the validity of the test expression $q_{2m+1} = pr^m + s$. For $m = 0$ we have that it is true because

$$pr^0 + s = p + s = q_0 A - s \cdot BA + s = q_0 A - sr + s = q_0 A + s\left(1-r\right) = q_0 A + F = q_1 \tag{55}$$

Classically, it is supposed that $q_{2m+1} = pr^m + s$ is true. So

$$q_{2m+2} = q_{2(m+1)} = q_{2m+1}B = pr^m B + sB \tag{56}$$

therefore

$$q_{2(m+1)+1} = pr^m \cdot BA + s \cdot BA + F = pr^{m+1} + sr + F = pr^{m+1} + sr + s - sr = pr^{m+1} + s. \tag{57}$$

The previous demonstration made by the induction method on m of the formula for $q_{2m+1} = pr^m + s$ proves its validity. Therefore the main result is given by

$$q_{n=2m+1} = pr^m + s, m = 0,1,2,\cdots \quad p = \left(q_0 - sB\right)A \quad 0 < s = \frac{F}{1-r} < 1 \tag{58}$$

because the others data can be calculate from this expression which in turn depends on the initial condition q_0 contained in $p = \left(q_0 - sB\right)A$. It is clear that $s = \frac{F}{1-r}$ is independent of the initial condition q_0. The cases of $n = 2,4,6,\cdots$ are given by

$$q_{n=2(m+1)} = pBr^m + sB \quad m = 0,1,2,\cdots \quad 0 < sB < s < 1 \tag{59}$$

According to these equations, all the orbits will tends to

$$\begin{aligned} \lim_{m\to\infty} q_{2m+1} &= q_a^\omega = s \\ \lim_{m\to\infty} q_{2(m+1)} &= q_b^\omega = sB \end{aligned} \tag{60}$$

That's great! The results allow concluding that there is a limit cycle symbolized by ω which, according to the calculations made, is the trajectory illustrated in **Figure 4(a)**

$$q_{2m+1} = s \quad q_{2(m+1)} = sB. \tag{61}$$

Note also that if the initial condition q_0 is such that

$$p = \left(q_0 - sB\right)A = 0,$$

equivalently to

$$q_0 = sB \tag{62}$$

the trajectory in the phase space it will correspond precisely to the trajectory known (in Dynamical system) as limit cycle ω of the differential equations. The values of the charges q_n and currents $\dot{q}_n$ will not be different—respectively—of

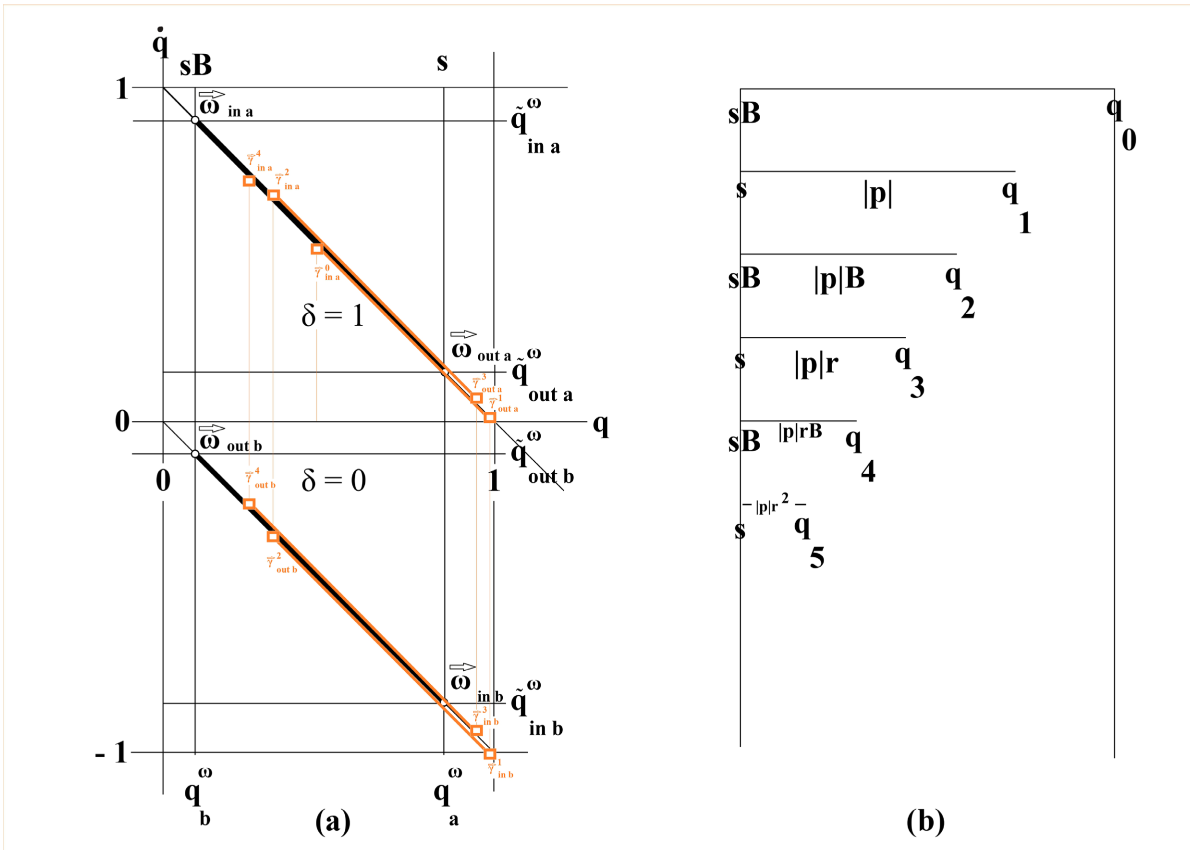

Figure 4. (a) ω-limit cycle. (b) Absolute value of a_n.

the charge q_{n+1} and current $\dot{q}_{n+1}$. In other words, the output signal is periodical of time's period T

$$\begin{aligned} q_{n=2m+1} &= q_a^\omega = s \\ q_0 = q_{n=2(m+1)} &= q_b^\omega = q_a^\omega B \end{aligned} \tag{63}$$

which is known in Control systems or Control theory as a case of the steady state response of the two-pole switch RC circuit.

Replacing $q_a^\omega = s$ and $q_b^\omega = sB$ in—respectively—in q_{2m+1} and $q_{2(m+1)}$ results

$$\begin{aligned} &q_{2m+1} = pr^m + q_a^\omega, m = 0,1,2,\cdots \quad p = \left(q_0 - q_b^\omega\right)A \quad 0 < q_a^\omega = \frac{F}{1-r} < 1 \\ &q_{2(m+1)} = pBr^m + q_b^\omega \quad m = 0,1,2,\cdots \quad 0 < q_b^\omega < q_a^\omega < 1 \end{aligned} \tag{64}$$

If γ represents any trajectory such that $\gamma \neq \omega$ ($p \neq 0$), the possible trajectories can be expressed as:

$$a_{2m+1} = \frac{q_{2m+1} - q_a^\omega}{p} = r^m = \frac{q_{2(m+1)} - q_b^\omega}{pB} = a_{2(m+1)} \tag{65}$$

where a_{2m+1} and $a_{2(m+1)}$ can be interpreted as the distances relative to the limit path ω and which form geometric progressions converging to 0, with the same ratio r, increasing if $p < 0$ and decreasing if $p > 0$ (see Figure 4(b)).

With a little more work it is possible to include the time, initial condition, time's intervals a,b and the data of the limit cycle ω:

$$q_n = \begin{Bmatrix} q_0 & . \\ q_{2m+1} = pr^m + q_a^\omega & m = 0,1,2,\cdots \\ q_{2(m+1)} = pBr^m + q_b^\omega & . \end{Bmatrix} \quad p = \left(q_0 - q_b^\omega\right)A$$
$$t_n < t = t_n + x < t_{n+1} = t_n + \theta \quad q = q_n \mathrm{e}^{-x} + \delta\left(1 - \mathrm{e}^{-x}\right) \quad \boldsymbol{r} = (1,-1)q + (0,1)\delta \tag{66}$$

Based on the definitions of quantities $q_{in\,\theta}^n, q_{out\,\theta}^n, \tilde{q}_{in\,\theta}^n, \tilde{q}_{out\,\theta}^n$ are introduced

$$\begin{aligned} \boldsymbol{r}_{in\,\theta}^n &= \lim_{t \to t_n^+} \boldsymbol{r}(t) = \lim_{x \to 0^+} \boldsymbol{r} = \boldsymbol{r}(t_n +) = (1,-1)q_n + (0,1)\delta(t_n +) \\ \boldsymbol{r}_{out\,\theta}^{n+1} &= \lim_{t \to t_{n+1}^-} \boldsymbol{r}(t) = \lim_{x \to \theta^-} \boldsymbol{r} = \boldsymbol{r}(t_{n+1} -) = (1,-1)q_{n+1} + (0,1)\delta(t_{n+1} -) \end{aligned} \tag{67}$$

useful to location the trajectory. This way the location of the ω-trajectory is given by

$$\begin{gathered} \boldsymbol{\omega}_{in\,a} = \boldsymbol{\omega}_{out\,b} + (0,1) \quad \boldsymbol{\omega}_{out\,a} = \boldsymbol{\omega}_{in\,b} + (0,1) \\ \boldsymbol{\omega}_{out\,b} = (1,-1)sB \qquad \boldsymbol{\omega}_{in\,b} = (1,-1)s \\ t_n < t = t_n + x < t_{n+1} = t_n + \theta \quad q = q_n \mathrm{e}^{-x} + \delta\left(1 - \mathrm{e}^{-x}\right) \quad \boldsymbol{r} = (1,-1)q + (0,1)\delta \end{gathered} \tag{68}$$

This means that ω-trajectory are the dark lines joining $\boldsymbol{\omega}_{in\,a}$ with $\boldsymbol{\omega}_{out\,a}$ and $\boldsymbol{\omega}_{in\,b}$ with $\boldsymbol{\omega}_{out\,b}$. To any γ-trajectory corresponds the points

$$\begin{gathered} \boldsymbol{\gamma}_{in\,a}^{2m} = \boldsymbol{\omega}_{in\,a} + (1,-1)\begin{Bmatrix} q_0 - q_b^\omega \\ pBr^m \end{Bmatrix} \begin{Bmatrix} m = 0 \\ m \ge 1 \end{Bmatrix} \quad \boldsymbol{\gamma}_{out\,a}^{2m+1} = \boldsymbol{\omega}_{out\,a} + (1,-1)pr^m \\ \boldsymbol{\gamma}_{out\,b}^{2(m+1)} = \boldsymbol{\omega}_{out\,b} + (1,-1)pBr^m \quad \boldsymbol{\gamma}_{in\,b}^{2m+1} = \boldsymbol{\omega}_{in\,b} + (1,-1)pr^m \\ t_n < t = t_n + x < t_{n+1} = t_n + \theta \quad q = q_n \mathrm{e}^{-x} + \delta\left(1 - \mathrm{e}^{-x}\right) \quad \boldsymbol{r} = (1,-1)q + (0,1)\delta \end{gathered} \tag{69}$$

between $\boldsymbol{\gamma}_{in\,\theta}^n$ and $\boldsymbol{\gamma}_{out\,\theta}^{n+1}$. Orange lines in **Figure 4** are elements of paths with positive p.

3. Conclusions

The whole work on the responses of a simple circuit where a resistor, a capacitor and a square-wave-type voltage source are in series (GE's RC circuit) can be summarized graphically in **Figure 4** and analytically in Equations (29), (31) or (68), (69).

Calculations in the GE have proven the existence of the ω-limit cycle via Equations (59)-(61) and the dark lines path of **Figure 4(a)** joining $\boldsymbol{\omega}_{in\,\theta}$ with $\boldsymbol{\omega}_{out\,\theta}$ where $\theta = a,b$. During the ω-limit cycle a graph of the charge on the capacitor q versus time t is a periodical function of period $T = a + b$. When the switch is moved to $\delta = 1$ the charge on the capacitor increases exponentially from q_b^ω until q_a^ω (charging capacitor) and when the switch is moved to $\delta = 0$ the charge on the capacitor decreases exponentially from q_a^ω until q_b^ω (discharging capacitor) and so on. Note that the ω-limit cycle is independent of the initial charge stored in the capacitor. But if the initial condition is such that the Equation (62) is verified then the orbit in the phase space will be just the ω-limit cycle. Moreover, when the initial condition does not correspond to the ω-limit cycle, then the path will be similarly located on the lines $\delta = 1$, $\delta = 0$ and will tend to the ω-limit cycle according to Equation (66), (69) in geometric progression and according to Figure 4(b).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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