An Efficient Method to Find Approximate Solutions for Emden-Fowler Equations of nth Order

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Abstract

In this research, Emden-Fowler equations of higher order with boundary conditions are considered and solved using Modified Adomian Decomposition Method (MADM). We defined a new differential operator under two conditions: first condition when $m \leq 0$ and second condition when $m \geq 0$. From this operator, we got three types of Emden-Fowler equations of higher order. The new method is evaluated by using many examples, the results obtained through this method reveal the effectiveness of this method for these type of equations, especially when comparisons are made with the exact solution.

1 Introduction

We consider the Emden-Fowler equation of the type [1,2]

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$$y'' + \frac{m+n}{x}y' + \frac{m(n-1)}{x^2}y + g(x,y) = 0, \tag{1}$$

where $g(x,y)$ is a known function of (x,y), $n \geq 1$ and m will take two values when $m \geq 0$ under specific conditions and when $m \leq 0$ under specific conditions. Those conditions will be discussed in details later. Such issues frequently appear usually in numerous ranges of science and engineering. For instance, the emerge in fluid mechanics, quantum mechanics, chemical rector hypothesis, geographics, and forth. If we examine (1), we can say that the main difficulty lies in the singular behaviour that occurs at $x = 0$. The Emden-Fowler equation is regarded to be of great importance in mathematics. Due to its great significance, several method were introduced to study this equation such as Ramos [3] finding a series of solutions to nonlinear equations to solve the Lane-Emden equation using Homotopy Perturation Method.

Wazwaz [4], Wazwaz and Rach [5], Wazwaz [6] had solved the Lane-Emden-Fowler equation using Variational Iteration Method. In addition, there are other methods that have been given to solve Emden-Fowler equation. For example, Chebyshev Neural Method by Mall and Chakraverty [7], Haar Wavelet Quasilinearization Method by Singh et al. [8]. Radial Basis Function Collocation Method, Radial Basis Function Differential Quadrature Method by Parand et al. [9]. Finite Element Method, Adomian Decomposition Algorithm by Hosson [10] and Adomian Decomposition Method [11,12,13]. In [11] Wazwaz et al. had solved three types of Emden-Fowler equations of fourth order with initial conditions. In [12] he had presented a reliable algorithm to determine the solution of the generalized Emden-Fowler equation. In our study, we studied three types of Emden-Fowler equations of higher order with boundary conditions using MADM.

The Adomain decomposition method (ADM) [14,15,16,17] has appeared in 1980s and was firstly introduced by the American scientist Adomain George. This method has been proven to be efficient and reliable in solving different attained by (ADM) converge to the exact solution. This method has attracted the attention of various researchers and therefore was used by many mathematicians to solve different kinds of equations, introducing many modifications on it seen in Biazar and Hosseini [18], Hasan and Zhu [19]. We object in this study to solve various kinds of Emden-Fowler type equations of higher order. We proposed a highly effective differential operator to solve different types of Emden-Fowler equations.

2 Structure of Emden-Fowler Kinds Equations

It is significant to mention that the Emden-Fowler equation (1) was derived by using the equation

$$x^{-n}\frac{d}{dx}x^{n-m}\frac{d}{dx}x^m(y) + g(x,y) = 0. \tag{2}$$

The sense of (2) is used in order to derive the Emden-Fowler equations of different order.

$$x^{-n}\frac{d}{dx}x^{n-m}\frac{d}{dx}x^m\frac{d^{k-1}}{dx^{k-1}}(y) + g(x,y) = 0, \tag{3}$$

where $n \geq 1$. To determine such different equations of higher order we set m to different values.

2.1 First kind for $m \neq 0, n \neq 1$,

$$y^{(k+1)} + \frac{m+n}{x}y^k + \frac{m(n-1)}{x^2}y^{(k-1)} + g(x,y) = 0, \tag{4}$$

2.2 Second kind for m=0,

put $m = 0$ in eq.(3) obtains

$$y^{(k+1)} + \frac{n}{x}y^k + g(x,y) = 0 \tag{5}$$

2.3 Third kind for $m = -n$,

put $m = -n$, in eq.(3) obtains

$$y^{(k+1)} - \frac{n(n-1)}{x^2}y^{(k-1)} + g(x,y) = 0, \tag{6}$$

3   Description of the Method and Its Application

Assume the singular boundary value problem of higher order ordinary differential equations as eq.(4)
under the two conditions

1. When m$\leq$ 0, we use the following condition

$$y(a_0) = A, y'(a_1) = B, y''(a_2) = C, ..., y^{(k-1)}(a_n) = D, y^{(k)}(0) = E, \tag{7}$$

where $a_n \neq 0$

2. When m$\geq$ 0. we use the following condition

$$y(a_0) = A, y'(a_1) = B, y''(a_2) = C, ..., y^{(k-2)}(a_n) = D,$$

$$y^{(k)}(0) = E, y^{(k-1)}(0) = F, \tag{8}$$

where $g(x,y)$ is a known function and A, B, C, D, E, F are constants and $n \geq 1, k \geq 1$.

We offer the new differential operator as follows

$$L(.) = x^{-n} \frac{d}{dx} x^{n-m} \frac{d}{dx} x^m \frac{d^{(k-1)}}{dx^{(k-1)}}(.). \tag{9}$$

Eq.(4) can be written as

$$Ly = -g(x,y). \tag{10}$$

For the conditions (7),(8) we have the inverse operator L^{-1} respectively

$$L^{-1}(.) = \underbrace{\int_{a_0}^x \int_{a_1}^x \int_{a_2}^x ... \int_{a_{n-1}}^x}_{(k-1)} x^{-m} \int_{a_n}^x x^{m-n} \int_0^x x^n \underbrace{dxdxdx...dxdxdx}_{(k+1)}, \tag{11}$$

$$L^{-1}(.) = \underbrace{\int_{a_0}^x \int_{a_1}^x \int_{a_2}^x ... \int_{a_n}^x}_{(k-1)} x^{-m} \int_0^x x^{m-n} \int_0^x x^n \underbrace{dxdxdx...dxdxdx}_{(k+1)}. \tag{12}$$

By applying L^{-1} on (10), we have

$$y(x) = \gamma(x) - L^{-1}(g(x,y), \tag{13}$$

such that

$$L(\gamma(x)) = 0.$$

The method by Adomian is given the solution $y(x)$ and the function $g(x,y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{14}$$

and

$$g(x, y) = \sum_{n=0}^{\infty} A_n, \tag{15}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen [14,16] to formulate Adomian polynomials. The following algorithm:

$$A_0 = G(y_0),$$

$$A_1 = y_1 G'(y_0),$$

$$A_2 = y_2 G'(y_0) + \frac{1}{2!} y_1^2 G''(y_0),$$

$$A_3 = y_3 G'(y_0) + y_1 y_2 G''(y_0) + \frac{1}{3!} y_1^3 G'''(y_0), \tag{16}$$

....

Can be used to build Adomian polynomials, when $G(y)$ is any function. From (15),(14) and (13) we have

$$\sum_{n=0}^{\infty} y_n(x) = \gamma(x) + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{17}$$

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$y_0 = \gamma(x),$$

$$y_{(n+1)} = L^{-1} A_n, \ n \geq 0, \tag{18}$$

thus

$$y_0 = \gamma(x),$$
$$y_1 = L^{-1} A_0,$$
$$y_2 = L^{-1} A_1,$$
$$y_3 = L^{-1} A_2, \tag{19}$$

....

Using the equations (16) and (19) we can determine the components y_n and therefore we can immediately obtain series solutions of $y(x)$ in (17). In addition, and for numerical reasons, we can use the n-term approximate

$$\psi_n = \sum_{n=0}^{n-1} y_n, \tag{20}$$

in order to approximate the exact solution. The validity of the above presented approach can be achieved through testing it on various types of several linear and non-linear differential equations with initial value problems.

3.1 Examples on the first type of Emden-Fowler equations of nth order

In this section, we study many examples for different values of m. In example (1) we study the equation of second order when $m = -3, n = 2$, in examples (2,3) we study the equation of third order when $m = 4, m = -4$, respectively and $n = 6$, also in examples (4,5) we study the equation of fifth order when $m = 3, m = -3$, respectively and $n = 2$.

Example.1 Substitute $m = -3, n = 2, k = 1$, in equation (4) we get

$$y'' - \frac{1}{x} y - \frac{3}{x^2} y - x^6 + y^2 = 0, \tag{21}$$

$$y(1) = 1, y'(0) = 0.$$

Exact solution is $y(x) = x^3$.

Eq.(21) can be written as

$$Ly = x^6 - y^2, \tag{22}$$

where

$$L(.) = x^{-2} \frac{d}{dx} x^5 \frac{d}{dx} x^{-3}(.),$$

and

$$L^{-1} = x^3 \int_1^x x^{-5} \int_0^x x^2(.) dx dx,$$

applying L^{-1} on (22) we find

$$y = x^3 + L^{-1} x^6 - L^{-1} y^2,$$

therefore

$$y = 0.977778 \, x^3 + 0.0222222 \, x^8 - L^{-1} y^2. \tag{23}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (23) gives

$$\sum_{n=0}^{\infty} y_n(x) = 0.977778 \, x^3 + 0.0222222 \, x^8 - L^{-1} A_n, \tag{24}$$

$$y_0 = 0.977778 \, x^3 + 0.0222222 \, x^8,$$

$$y_{n+1} = -L^{-1} A_n, n \geq 0, \tag{25}$$
$$A_0 = y_0^2,$$

$$A_1 = 2y_0 y_1,$$

$$A_2 = y_1^2 + 2y_0 y_2,$$

$$A_3 = 2y_1 y_2 + 2y_0 y_3, \tag{26}$$

$$....$$

From (25) and (26) we get

$$y_0 = 0.9777777778 x^3 + 0.02222222222 x^8,$$

$$y_1 = 0.0215577 \, x^3 - 0.0212455 \, x^8 - 0.000310406 \, x^{13} - 1.73273 \, 10^{-6} \, x^{18},$$

$$y_2 = -0.000641429 \, x^3 + 0.000936828 \, x^8 - 0.000289919 \, x^{13} - 5.44302 \, 10^{-6} \, x^{18}$$
$$- 3.58005 \, 10^{-8} \, x^{23} - 1.06221 \, 10^{-10} \, x^{28},$$

$$y_3 = -0.0000222338 \, x^3 + 0.0000382019 \, x^8 - 0.0000194251 \, x^{13} + 3.38002 \, 10^{-6} \, x^{18}$$
$$+ 7.63421 \, 10^{-8} \, x^{23} + 6.64689 \, 10^{-10} \, x^{28} + 2.81819 \, 10^{-12} \, x^{33} + 5.65807 \, 10^{-15} \, x^{38},$$

$$y(x) = y_0 + y_1 + y_2 + y_3 = 0.998716 \, x^3 + 0.00187531 \, x^8 - 0.0005809 \, x^{13} - 0.0000105558 \, x^{18}$$
$$- 1.12143 \, 10^{-7} \, x^{23} - 7.70909 \, 10^{-10} \, x^{28} - 2.81819 \, 10^{-12} \, x^{33} - 5.65807 \, 10^{-15} \, x^{38},$$

Table 1 and Fig. 1 explain the convergence between (MADM) and the exact solution

Table 1. Comparison of numerical errors between the right solution $y = x^3$ and the MADM solution $y = \sum_{n=0}^{3} y_n(x)$.

| x	Exact	MADM	Absolute error
0.0	0.000	0.000000000	0.000000000
0.1	0.001	0.000998716	0.000001284
0.2	0.008	0.007989730	0.000010270
0.3	0.027	0.026965500	0.000034500
0.4	0.064	0.063919100	0.000080900
0.5	0.125	0.124847000	0.000153000
0.6	0.216	0.215753000	0.000247000
0.7	0.343	0.342662000	0.000338000
0.8	0.512	0.511625000	0.000375000
0.9	0.729	0.728722000	0.000278000
1.0	1.000	1.000000000	0.000000000

Exact ——— MADM
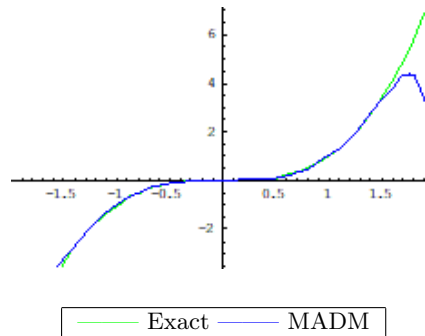
Fig. 1. Comparison of exact and approximate solution curves for Example 1

Example 2. Consider equation:

$$y''' + \frac{10}{x} y'' + \frac{20}{x^2} y' - 224x + e^{x^4} - e^y = 0, \tag{27}$$

with the conditions

$$y(\frac{1}{2}) = \frac{1}{16}, y'(0) = 0, y''(0) = 0,$$

when $m = 4, n = 6, k = 2$ in (4),
in ADM operator form equation (27) becomes

$$Ly = 224x - e^{x^4} + e^y, \tag{28}$$

where

$$L(.) = x^{-6} \frac{d}{dx} x^2 \frac{d}{dx} x^4 \frac{d}{dx}(.),$$

so, L^{-1} is given by

$$L^{-1}(.) = \int_{\frac{1}{2}}^{x} x^{-4} \int_{0}^{x} x^{-2} \int_{0}^{x} x^6 (.) dx dx.$$

Taking L^{-1} to both side of (28) and using the conditions we obtain

$$y(x) = L^{-1}(224x - e^{x^4}) + L^{-1}e^y, \tag{29}$$

Displace the decomposition series $\sum_{n=0}^{\infty} y_n(x)$ into (29) gives

$$\sum_{n=0}^{\infty} y_n(x) = 0.00100232 - 0.00793651\, x^3 + x^4 - 0.0012987\, x^7 - 0.00021645\, x^{11}$$

$$+ L^{-1}A_n, \tag{30}$$

the ADM introduce the recursive relation

$$y_0 = 0.00100232 - 0.00793651\, x^3 + x^4 - 0.0012987\, x^7 - 0.00021645\, x^{11},$$

$$y_{n+1} = L^{-1}(A_n), n \geq 0, \tag{31}$$

then

$$y_0 = 0.00100232 - 0.00793651\, x^3 + x^4 - 0.0012987\, x^7 - 0.00021645\, x^{11},$$

$$y_1 = -0.00100309 + 0.00794447\, x^3 - 0.000014712\, x^6 + 0.0013\, x^7 + 2.24542\, 10^{-8}\, x^9$$

$$-5.07938\, 10^{-6}\, x^{10} + 0.000216667\, x^{11} - 2.89588\, 10^{-11}\, x^{12},$$

$$y_2 = 7.71219\, 10^{-7} - 7.96898\, 10^{-6}\, x^3 + 0.0000147415\, x^6 - 1.30401\, 10^{-6}\, x^7 - 5.5465\, 10^{-8}\, x^9$$

$$+5.08957\, 10^{-6}\, x^{10} - 2.17336\, 10^{-7}\, x^{11} + 1.3538\, 10^{-10}\, x^{12},$$

$$...$$

Thus, the approximate solution is

$$y(x) = y_0 + y_1 + y_2 = 8.70116\, 10^{-10} - 1.01093\, 10^{-8}\, x^3 + x^4 + 2.95108\, 10^{-8}\, x^6 - 1.65425\, 10^{-9}\, x^7$$

$$-3.30109\, 10^{-8}\, x^9 + 1.01887\, 10^{-8}\, x^{10} - 2.75709\, 10^{-10}\, x^{11} + 1.06421\, 10^{-10}\, x^{12}.$$

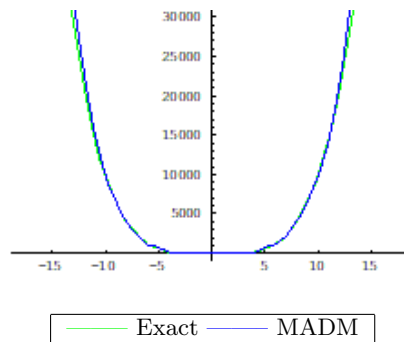Fig. 2 shows a good approximate between MADM solution and exact solution.

Fig. 2. Comparison of exact and approximate solution curves for Example 2

Example 3. Consider equation:

$$y''' + \frac{2}{x}y'' - \frac{20}{x^2}y' + 32x + e^{x^4} - e^y = 0, \tag{32}$$

with the conditions

$$y(\frac{1}{2}) = \frac{1}{16}, y'(\frac{1}{3}) = \frac{4}{27}, y''(0) = 0,$$

when $m = -4, n = 6, k = 2$ in (4),
rewrite eq.(32) as follows

$$Ly = -32x - e^{x^4} + e^y, \tag{33}$$

where

$$L(.) = x^{-6} \frac{d}{dx} x^{10} \frac{d}{dx} x^{-4} \frac{d}{dx}(.),$$

so, L^{-1} is given by

$$L^{-1}(.) = \int_{\frac{1}{2}}^{x} x^4 \int_{\frac{1}{3}}^{x} x^{-10} \int_{0}^{x} x^6 (.) dx dx dx.$$

Taking L^{-1} to both side of (33) and using the conditions we obtain

$$y(x) = L^{-1}(-32x - e^{x^4}) + L^{-1}e^y, \tag{34}$$

substituting the decomposition series $\sum_{n=0}^{\infty} y_n(x)$ into (34) gives

$$\sum_{n=0}^{\infty} y_n(x) = 0.00106103 + 0.0238095\,x^3 + x^4 - 0.12756\,x^5 - 0.00649351\,x^7$$

$$- 0.000505051\,x^{11} + L^{-1}A_n, \tag{35}$$

the ADM introduce the recursive relation

$$y_0 = 0.00106103 + 0.0238095\,x^3 + x^4 - 0.12756\,x^5 - 0.00649351\,x^7 - 0.000505051\,x^{11},$$

$$y_{n+1} = L^{-1}(A_n), n \geq 0, \tag{36}$$

$$y_0 = 0.00106103 + 0.0238095\,x^3 + x^4 - 0.12756\,x^5 - 0.00649351\,x^7 - 0.000505051\,x^{11},$$

$$y_1 = -0.00106249 - 0.0238348\,x^3 + 0.\,x^4 + 0.127562\,x^5 + 0.000397247\,x^6 + 0.0065004\,x^7$$

$$-0.000443386\,x^8 + 6.06298\,10^{-7}\,x^9 + 0.0000247634\,x^{10} + ... + 0.\,x^5\,\log(x),$$

$$y_2 = 1.45763\,10^{-6} + 0.0000253241\,x^3 + 0.\,x^4 - 2.64701\,10^{-6}\,x^5 - 0.00039809\,x^6 - 6.90658\,10^{-6}\,x^7$$

$$+0.000443867\,x^8 - 3.64809\,10^{-7}\,x^9 - 0.000024816\,x^{10} + ... + 0.\,x^{18}\,\log(x),$$

The approximate solution by MADM is given by

$$y(x) = y_0 + y_1 + y_2 = 2.79822\,10^{-9} + 4.80873\,10^{-8}\,x^3 + 1.\,x^4 + 4.01243\,10^{-8}\,x^5 - 8.43783\,10^{-7}\,x^6$$

$$-1.31147\,10^{-8}\,x^7 + 4.80432\,10^{-7}\,x^8 + 2.41489\,10^{-7}\,x^9 - 5.25995\,10^{-8}\,x^{10} + ... + 0.\,x^{18}\,\log(x).$$

Example 4. Substitute $m = 3, n = 2, k = 4$, in equation (4) we get:

$$y^{(5)} + \frac{5}{x}y^{(4)} + \frac{3}{x^2}y^{(3)} - \frac{24 - 36x}{x^2}e^{-5y} = 0, \tag{37}$$

with the conditions

$$y(0) = \log 2, y'(0) = \frac{1}{2}, y''(\frac{1}{2}) = \frac{-2}{25}, y'''(0) = \frac{1}{4}, y''''(0) = \frac{-3}{8},$$

in an oprator form eq.(37) can be written as

$$Ly = \frac{24 - 36x}{x^2}e^{-5y}, \tag{38}$$

where

$$L(.) = x^{-2} \frac{d}{dx} x^{-1} \frac{d}{dx} x^3 \frac{d^3}{dx^3}(.),$$

so, L^{-1} is given by

$$L^{-1}(.) = \int_0^x \int_0^x \int_{\frac{1}{2}}^x x^{-3} \int_0^x x \int_0^x x^2(.)dxdx.$$

Taking L^{-1} (38) and using the conditions we obtain

$$y(x) = 0.693147 + 0.5\,x - 0.08\,x^2 + L^{-1}(\frac{24 - 36x}{x^2}e^{-5y}), \tag{39}$$

Replace the decomposition series $\sum_{n=0}^{\infty} y_n(x)$ into (39) gives

$$\sum_{n=0}^{\infty} y_n(x) = 0.693147 + 0.5\,x - 0.08\,x^2 + L^{-1}(\frac{24 - 36x}{x^2}A_n), \tag{40}$$

the ADM introduce the repetitive relation

$$y_0 = 0.693147 + 0.5\,x - 0.08\,x^2,$$
$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{41}$$

$$y_0 = 0.693147 + 0.5\,x - 0.08\,x^2,$$
$$y_1 = -0.0449217\,x^2 + 0.0416667\,x^3 - 0.015625\,x^4 + 0.0060625\,x^5 - 0.00231554\,x^6 + 0.000853455\,x^7$$
$$-0.000301892\,x^8 + 0.000102371\,x^9 - 0.0000333048\,x^{10} + ... + 9.09044\,10^{-7}\,x^{13},$$
$$y_2 = 0.0000778897\,x^2 + 0.000187174\,x^5 - 0.000288221\,x^6 + 0.000259743\,x^7 - 0.000179296\,x^8$$
$$+0.000104691\,x^9 - 0.0000542495\,x^{10} + ... + 4.65124\,10^{-6}\,x^{13},$$

$$...$$

The approximate solution by MADM is given by

$$y(x) = y_0 + y_1 + y_2 = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.0416667\,x^3 - 0.015625\,x^4 + 0.00624967\,x^5$$

$$-0.00260376\,x^6 + 0.0011132\,x^7 - 0.000481188\,x^8 + 0.000207063\,x^9 - 0.0000875543\,x^{10}$$

$$+... + 5.56029\,10^{-6}\,x^{13}$$

Take notice, the exact solution $y(x) = \ln(x+2)$ can be written in a series form as

$$y(x) = 0.693147 + 0.5x - 0.125\,x^2 + 0.0416667\,x^3 - 0.015625\,x^4 + 0.00625\,x^5$$

$$-0.00260417\,x^6 + 0.00111607\,x^7 - 0.000488281\,x^8 + 0.000217014\,x^9 - 0.0000976563\,x^{10}$$

$$+... + 9.39002\,10^{-6}\,x^{13}.$$

Example 5. Substitute $m = -3, n = 2, k = 4$, in equation (4) we obtain:

$$y^{(5)} - \frac{1}{x}y^{(4)} - \frac{3}{x^2}y^{(3)} - \frac{12(-2 - x + 2x^2)}{x^2}e^{-5y} = 0, \tag{42}$$

with the conditions

$$y(0) = \log 2, y'(0) = \frac{1}{2}, y''(0) = \frac{-1}{4}, y'''(1) = \frac{2}{27}, y''''(0) = \frac{-3}{8},$$

note that the exact solution is $\ln(x+2)$,
in an oprator form eq.(42) can be written as

$$Ly = \frac{12(-2 - x + 2x^2)}{x^2}e^{-5y}, \tag{43}$$

where

$$L(.) = x^{-2} \frac{d}{dx} x^5 \frac{d}{dx} x^{-3} \frac{d^3}{dx^3}(.),$$

so, L^{-1} is given by

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x x^3 \int_1^x x^{-5} \int_0^x x^2 (.) dx dx.$$

Taking L^{-1} to (43) and using the conditions we obtain

$$y(x) = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.000617284\,x^6$$

$$+ L^{-1}\left(\frac{12(-2 - x + 2x^2)}{x^2} e^{-5y}\right), \tag{44}$$

Replace the decomposition series $\sum_{n=0}^{\infty} y_n(x)$ into (44) have

$$\sum_{n=0}^{\infty} y_n(x) = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.000617284\,x^6$$

$$+ L^{-1}\left(\frac{24 - 36x}{x^2} A_n\right), \tag{45}$$

the ADM introduce the recursive relation

$$y_0 = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.000617284\,x^6,$$

$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{46}$$

$$y_0 = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.000617284\,x^6,$$
$$y_1 = 0.0416667\,x^3 - 0.015625\,x^4 + 0.00625\,x^5 - 0.00346015\,x^6 + 0.00146949\,x^7 - 0.000581287\,x^8$$
$$+0.00022935\,x^9 - 0.0000878734\,x^{10} + ...,$$
$$y_2 = -0.000353423\,x^7 + 0.000093006\,x^8 + x^9 \left(-0.0000107847 - 1.15335\,10^{-7}\,\log(x)\right)$$
$$+x^{10} \left(-0.0000116657 + 1.05964\,10^{-7}\,\log(x)\right) + x^6 \left(0.000189275 + 0.000325521\,\log(x)\right) + ...,$$

$$...$$

The approximate solution by MADM is given by

$$y(x) = y_0 + y_1 + y_2 = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.0416667\,x^3 - 0.015625\,x^4 + 0.00625\,x^5$$

$$-0.00265359\,x^6 + 0.00111607\,x^7 - 0.000488281\,x^8 + x^9 \left(0.000218566 - 1.15335\,10^{-7}\,\log(x)\right) + ...$$

Take notice, exact solution $y(x) = \ln(x + 2)$ can be written in a series form as

$$y(x) = 0.693147 + 0.5\,x - 0.125\,x^2 + 0.0416667\,x^3 - 0.015625\,x^4 + 0.00625\,x^5$$

$$-0.00260417\,x^6 + 0.00111607\,x^7 - 0.000488281\,x^8$$

$$+0.000217014\,x^9 - 0.0000976563\,x^{10} + ...$$

As noticed in examples 4 and 5, the solution using ADM converges towards the exact solution with minor frequencies, which indicates the efficiency of the ADM as a method to solve those types of problems.
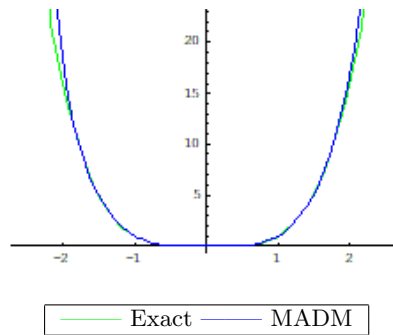
Fig. 3. Comparison of exact and approximate solution curves for Example 3

3.2 The second type of Emden-Fowler equations of nth order

The second Type of Emden-Fowler Equations of nth Order is

$$y^{(k+1)} + \frac{n}{x} y^{(k)} + g(x, y) = 0 \tag{47}$$

$$y(a_0) = A, y'(a_1) = B, y''(a_2) = C, ..., y^{(k-1)}(a_n) = D, y^{(k)}(0) = E.$$

Rewrite eq.(47) as follows

$$Ly = -g(x, y), \tag{48}$$

where

$$Ly = x^{-n} \frac{d}{dx} x^n \frac{d^k}{dx^k}(y),$$

and

$$L^{-1}(.) = \underbrace{\int_{a_0}^x \int_{a_1}^x \int_{a_2}^x ... \int_{a_{n-1}}^x \int_{a_n}^x}_{(k)} x^{-n} \int_0^x x^n \underbrace{dxdxdx...dxdxdx}_{(k+1)}.$$

By applying L^{-1} on (48) we have

$$y(x) = \gamma(x) - L^{-1} g(x, y), \tag{49}$$

where $\gamma(x)$ come out from using the conditions.

We will give thre examples on this kind of equations.

Example 6. Substitute $k = 2, n = 10$, in eq. (47) we have

$$y''' + \frac{10}{x} y'' - (1 + x^2 + \frac{20}{x} - y) = 0, \tag{50}$$

$$y(0) = 1, y'(1) = 2, y''(0) = 2.$$

And $y(x) = 1 + x^2$ is the exact solution.

Eq.(50) can be written as

$$Ly = 1 + x^2 + \frac{20}{x} - y, \tag{51}$$

where

$$L(.) = x^{-10} \frac{d}{dx} x^{10} \frac{d^2}{dx^2}(.),$$

and

$$L^{-1}(.) = \int_0^x \int_1^x x^{-10} \int_0^x x^{10}(.) dx dx dx.$$

Using L^{-1} on eq.(51), we get

$$y(x) = L^{-1}(1 + x^2 + \frac{20}{x}) - L^{-1}y. \tag{52}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (52) gives

$$\sum_{n=0}^{\infty} y(x) = L^{-1}(48 + x^8) - L^{-1}y_n. \tag{53}$$

$$y_0 = L^{-1}(1 + x^2 + \frac{20}{x}),$$

$$y_{n+1} = -L^{-1}(y_n), n \geq 0, \tag{54}$$

then

$$y_0 = 1 + x^2 - 0.0646853\,x + 0.0151515\,x^3 + 0.00384615\,x^5,$$

$$y_1 = 0.0631393\,x - 0.0151515\,x^3 + 0.000449204\,x^4 - 0.00384615\,x^5 - 0.000036075\,x^6 - 4.29258\,10^{-6}\,x^8,$$

$$y_2 = 0.00150778\,x - 0.000438467\,x^4 + 0.000036075\,x^6 - 7.13022\,10^{-7}\,x^7 + 4.29258\,10^{-6}\,x^8$$
$$+ 2.94731\,10^{-8}\,x^9 + 2.05387\,10^{-9}\,x^{11},$$

$$y_3 = 0.0000372946\,x - 0.0000104707\,x^4 + 6.9598\,10^{-7}\,x^7 - 2.94731\,10^{-8}\,x^9 + 4.40137\,10^{-10}\,x^{10}$$
$$- 2.05387\,10^{-9}\,x^{11} - 1.1164\,10^{-11}\,x^{12} - 5.12954\,10^{-13}\,x^{14},$$

$$y(x) = y_0 + y_1 + y_2 + y_3 = 1 + x^2 - 9.4724\,10^{-7}\,x + 2.65568\,10^{-7}\,x^4 - 1.70417\,10^{-8}\,x^7 + 4.40137\,10^{-10}\,x^{10}$$
$$- 1.1164\,10^{-11}\,x^{12} - 5.12954\,10^{-13}\,x^{14}.$$

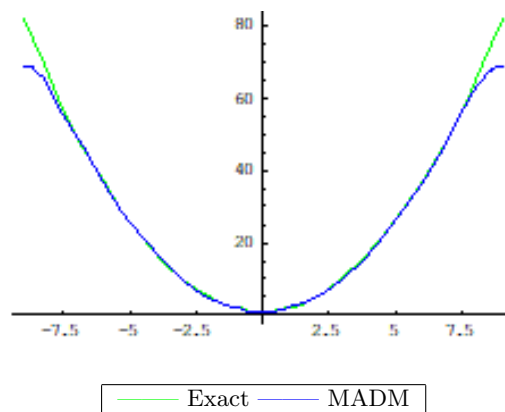Fig. 4 offer the comparison between MADM solution and exact solution.

Fig. 4. Comparison of exact and approximate solution curves for Example 6

Example 7. Substitute $k = 3, n = 1$, in eq. (47) we have

$$y^{(4)} + \frac{1}{x} y^{(3)} - 48 - x^8 + y^2 = 0, \tag{55}$$

$$y(1) = 1, y'(\frac{1}{2}) = \frac{1}{2}, y''(\frac{1}{6}) = \frac{1}{3}, y'''(0) = 0.$$

And $y(x) = x^4$ is the exact solution.

Eq.(55) can be written as

$$Ly = 48 + x^8 - y^2, \tag{56}$$

where

$$L(.) = x^{-1} \frac{d}{dx} x \frac{d^3}{dx^3}(.),$$

and

$$L^{-1}(.) = \int_1^x \int_{\frac{1}{2}}^x \int_{\frac{1}{6}}^x x^{-1} \int_0^x x(.)dxdxdxdx.$$

Using L^{-1} on eq.(56), we get

$$y(x) = L^{-1}(48 + x^8) - L^{-1}y^2. \tag{57}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (57) gives

$$\sum_{n=0}^{\infty} y(x) = L^{-1}(48 + x^8) - L^{-1}A_n. \tag{58}$$

$$y_0 = L^{-1}(48 + x^8),$$
$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{59}$$

where A_n are Adomian polynomials of nonlinear term y^2, as follows

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

$$A_2 = 2y_0y_2 + y_1^2, \tag{60}$$

$$...$$

From (59) and (60)

$$y_0 = -0.0000753137 - 4.43809\,10^{-7}\,x - 8.26909\,10^{-11}\,x^2 + x^4 + 0.0000757576\,x^{12},$$

$$y_1 = 0.0000752447 + 4.39215\,10^{-7}\,x + 5.75308\,10^{-11}\,x^2 - 1.1817\,10^{-10}\,x^4 - 3.71388\,10^{-13}\,x^5$$
$$-4.36296\,10^{-16}\,x^6 - 6.99028\,10^{-20}\,x^7 + 7.4716\,10^{-8}\,x^8 + ... - 1.12301\,10^{-14}\,x^{28},$$

$$y_2 = -6.88768\,10^{-8} - 4.55044\,10^{-9}\,x - 5.44271\,10^{-12}\,x^2 - 2.36123\,10^{-10}\,x^4 - 7.38591\,10^{-13}\,x^5$$
$$-8.56178\,10^{-16}\,x^6 - 1.17813\,10^{-19}\,x^7 + 7.46475\,10^{-8}\,x^8 + ... - 5.09824\,10^{-25}\,x^{44},$$

$$y(x) = y_0 + y_1 + y_2 = -1.37876\,10^{-7} - 9.14463\,10^{-9}\,x - 3.06027\,10^{-11}\,x^2 + 1.\,x^4 - 1.10998\,10^{-12}\,x^5$$
$$-1.29247\,10^{-15}\,x^6 - 1.87716\,10^{-19}\,x^7 + 1.49363\,10^{-7}\,x^8 + ... - 5.09824\,10^{-25}\,x^{44}.$$

Fig. 5 offer the comparison between MADM solution and exact solution.
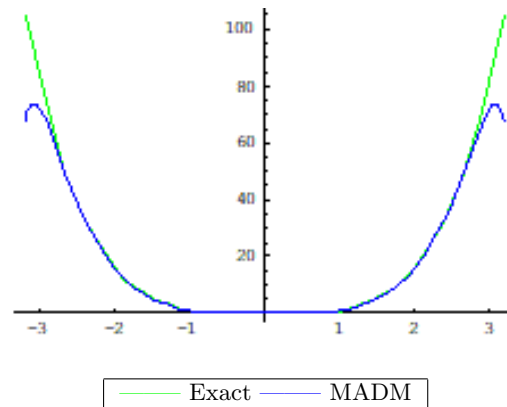
Fig. 5. Comparison of exact and approximate solution curves for Example 7

Example 8. We assume the Emden-Fowler type equation

$$y^{(6)} + \frac{3}{x}y^{(5)} - 16e^{x^2}(30 + 75x^2 + 36x^4 + 4x^6) = 0, \tag{61}$$

$$y(0) = 1, y'(0) = 0, y''(1) = 6e, y'''(\frac{1}{2}) = 7e^{\frac{1}{4}}, y''''(\frac{1}{3}) = \frac{1420}{81}e^{\frac{1}{9}}, y'''''(0) = 0.$$

Note that $y(x) = e^{x^2}$ is the exact solution.
In an operator form eq.(61) can be written as

$$Ly = 16e^{x^2}(30 + 75x^2 + 36x^4 + 4x^6). \tag{62}$$

Where

$$L(.) = x^{-3}\frac{d}{dx}x^3\frac{d^5}{dx^5}(.),$$

and

$$L^{-1}(.) = \int_0^x \int_0^x \int_1^x \int_{\frac{1}{2}}^x \int_{\frac{1}{3}}^x x^{-3} \int_0^x x^3(.)dxdxdxdxdxdx.$$

Using L^{-1} to both side of eq.(62), we have

$$y(x) = e^{x^2}, \tag{63}$$

example (8) shows the ability of our method in the finding of the exact solution.

3.3   The third Type of nth Order

The third Type of nth Order is

$$y^{(k+1)} - \frac{n(n-1)}{x^2}y^{(k-1)} + g(x, y) = 0 \tag{64}$$

$$y(a_0) = A, y'(a_1) = B, y''(a_2) = C, ..., y^{(k-1)}(a_n) = D, y^{(k)}(0) = E,$$

where $a_n \neq 0$. Eq.(59) can be written as

$$Ly = -g(x, y), \tag{65}$$

Where

$$L(.) = x^{-n} \frac{d}{dx} x^{2n} \frac{d}{dx} x^{-n} \frac{d^{(k-1)}}{dx^{(k-1)}}(.),$$

and inverse operator

$$L^{-1}(.) = \underbrace{\int_{a_0}^x \int_{a_1}^x \int_{a_2}^x \cdots \int_{a_{n-1}}^x}_{(k-1)} x^n \int_{a_n}^x x^{-2n} \int_0^x x^n(.) \underbrace{dxdxdx...dxdxdx}_{(k+1)}.$$

Applying L^{-1} on (60)

$$y(x) = \gamma(x) - L^{-1}g(x,y), \tag{66}$$

where $\gamma(x)$ come out from using the conditions.

we will study three examples on this kind for different order

Example 9. Substitute $k = 1, n = 2$, in eq. (64) we have

$$y'' - \frac{2}{x^2} y + (1 + \frac{2}{x^2}) \sin x = 0, \tag{67}$$

$$y(1) = \sin 1, y'(0) = 1.$$

Note that $y(x) = \sin x$ is the exact solution.

In an operator form eq.(67) can be written as

$$Ly = -(1 + \frac{2}{x^2}) \sin x. \tag{68}$$

Where

$$L(.) = x^{-2} \frac{d}{dx} x^4 \frac{d}{dx} x^{-2}(.),$$

and

$$L^{-1}(.) = x^2 \int_0^x x^{-4} \int_0^x x^2(.)dxdx.$$

Using L^{-1} to both side of eq.(68), we have

$$y(x) = \sin x, \tag{69}$$

in this example, we get the exact solution.

Example 10. Consider equation:

$$y''' - \frac{6}{x^2} y' + 12 + x^9 - y^3 = 0, \tag{70}$$

$$y(\frac{1}{3}) = \frac{1}{27}, y'(\frac{1}{2}) = \frac{3}{4}, y''(0) = 0.$$

When $k = 2, n = 3$, in eq.(64), and $y(x) = x^3$ is the exact solution.

We can write eq.(70) in an operator form as follows

$$Ly = -12 - x^9 + y^3, \tag{71}$$

where

$$L(.) = x^{-3} \frac{d}{dx} x^6 \frac{d}{dx} x^{-3} \frac{d}{dx}(.),$$

and

$$L^{-1}(.) = \int_{\frac{1}{3}}^{x} x^3 \int_{\frac{1}{2}}^{x} x^{-6} \int_{0}^{x} x^3(.)dxdxdx.$$

Applying L^{-1} to eq.(71), we get

$$y(x) = -1.14418\,10^{-7} + x^3 + 9.39002\,10^{-6}\,x^4 - 0.000801282\,x^{12} + L^{-1}y^3. \tag{72}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (72) gives

$$\sum_{n=0}^{\infty} y(x) = -1.14418\,10^{-7} + x^3 + 9.39002\,10^{-6}\,x^4 - 0.000801282\,x^{12}$$

$$+ L^{-1}A_n, \tag{73}$$

$$y_0 = -1.14418\,10^{-7} + x^3 + 9.39002\,10^{-6}\,x^4 - 0.000801282\,x^{12},$$
$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{74}$$

where A_n are Adomian polynomials of nonlinear term y^2, as follows

$$A_0 = y_0^3,$$
$$A_1 = 3y_0^2 y_1,$$
$$A_2 = 3y_0^2 y_2 + 3y_1^2 y_0, \tag{75}$$
$$...$$

Substituting (74) into (75) gives the components

$$y_0 = -1.14418\,10^{-7} + x^3 + 9.39002\,10^{-6}\,x^4 - 0.000801282\,x^{12},$$

$$y_1 = 1.14419\,10^{-7} + 1.24827\,10^{-22}\,x^3 - 9.39007\,10^{-6}\,x^4 + 4.67557\,10^{-16}\,x^6$$
$$+2.19518\,10^{-21}\,x^7 - 7.6279\,10^{-10}\,x^9 - 9.7672\,10^{-15}\,x^{10} - ... - 9.42244\,10^{-15}\,x^{39},$$
$$y_2 = -5.62238\,10^{-13} - 3.74482\,10^{-22}\,x^3 + 4.32737\,10^{-11}\,x^4 - 9.35118\,10^{-16}\,x^6 - 6.58558\,10^{-21}\,x^7$$
$$+7.62794\,10^{-10}\,x^9 + 1.95345\,10^{-14}\,x^{10} + ... - 6.61981\,10^{-26}\,x^{66},$$
$$y(x) = y_0 + y_1 + y_2 = -4.78541\,10^{-18} + 1.\,x^3 + 4.17001\,10^{-16}\,x^4 - 4.67561\,10^{-16}\,x^6 - 4.3904\,10^{-21}\,x^7$$
$$+3.74822\,10^{-15}\,x^9 + 9.76729\,10^{-15}\,x^{10} + ... - 6.61981\,10^{-26}\,x^{66},$$
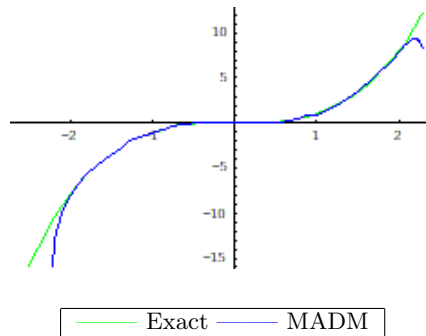

Fig. 6. Comparison of exact and approximate solution curves for Example 10

Example 11. Substitute $k = 7, n = 5$, in (64) we have

$$y^{(8)} - \frac{20}{x^2}y^{(6)} - (1 - \frac{20}{x^2})e^x + x - lny = 0, \tag{76}$$

$y(0) = 1, y'(0) = 1, y''(\frac{1}{3}) = e^{\frac{1}{3}}, y^{(3)}(\frac{1}{2}) = e^{\frac{1}{2}}, y^{(4)}(\frac{1}{4}) = e^{\frac{1}{4}}, y^{(5)}(\frac{1}{6}) = e^{\frac{1}{6}}, y^{(6)}(1) = e, y^{(7)}(0) = 1.$

And $y(x) = e^x$ is the exact solution.

We can write eq.(76) in an operator form as follows

$$Ly = (1 - \frac{20}{x^2})e^x - x + lny, \tag{77}$$

where

$$L(.) = x^{-5}\frac{d}{dx}x^{10}\frac{d}{dx}x^{-5}\frac{d^6}{dx^6}(.),$$

and

$$L^{-1}(.) = \int_0^x \int_0^x \int_{\frac{1}{3}}^x \int_{\frac{1}{2}}^x \int_{\frac{1}{4}}^x \int_{\frac{1}{6}}^x x^5 \int_1^x x^{-10} \int_0^x x^5(.)dxdxdxdxdxdxdxdx.$$

Applying L^{-1} to eq.(77), we get

$$y(x) = e^x + 1.15269\,10^{-6}\,x^2 - 1.1297\,10^{-6}\,x^3 - 1.26963\,10^{-10}\,x^4 - 1.12696\,10^{-7}\,x^5 + 1.18103\,10^{-6}\,x^9$$
$$- 2.14732\,10^{-7}\,x^{11} + L^{-1}lny. \tag{78}$$

Replace the decomposition series $y_n(x)$ for $y(x)$ into (78) gives

$$\sum_{n=0}^{\infty} y(x) = e^x + 1.15269\,10^{-6}\,x^2 - 1.1297\,10^{-6}\,x^3 - 1.26963\,10^{-10}\,x^4 - 1.12696\,10^{-7}\,x^5$$
$$+ 1.18103\,10^{-6}\,x^9 - 2.14732\,10^{-7}\,x^{11} + L^{-1}A_n. \tag{79}$$
$$y_0 = e^x + 1.15269\,10^{-6}\,x^2 - 1.1297\,10^{-6}\,x^3 - 1.26963\,10^{-10}\,x^4 - 1.12696\,10^{-7}\,x^5 + 1.18103\,10^{-6}\,x^9$$
$$+1.18103\,10^{-6}\,x^9 - 2.14732\,10^{-7}\,x^{11},$$
$$y_{n+1} = -L^{-1}(A_n), n \geq 0, \tag{80}$$

then

$$y_0 = e^x + 1.15269\,10^{-6}\,x^2 - 1.1297\,10^{-6}\,x^3 - 1.26963\,10^{-10}\,x^4 - 1.12696\,10^{-7}\,x^5 + 1.18103\,10^{-6}\,x^9$$
$$-2.14732\,10^{-7}\,x^{11},$$
$$y_1 = -1.15269\,10^{-6}x^2 + 1.1297\,10^{-6}\,x^3 + 1.26965\,10^{-10}\,x^4 + 1.12696\,10^{-7}\,x^5 - 1.18103\,10^{-6}\,x^9$$
$$-9.52949\,10^{-13}\,x^{10} + ... - 7.62383\,10^{-13}\,x^{11}\,\log(x),$$

$$y_2 = 1.76299\,10^{-13}x^2 - 1.72447\,10^{-13}\,x^3 - 1.41747\,10^{-15}\,x^4 - 1.50093\,10^{-14}\,x^5$$
$$+9.52949\,10^{-13}\,x^{10} + ... + 7.62383\,10^{-13}\,x^{11}\,\log(x),$$
$$y(x) = y_0 + y_1 + y_2 = e^x + 1.76299\,10^{-13}x^2 - 1.72447\,10^{-13}\,x^3 1.70921\,10^{-21}\,x^4 - 1.50093\,10^{-14}\,x^5$$
$$- 2.14732\,10^{-7}\,x^{11}, \tag{81}$$

Propagation equation (73) using Taylor series of order 10 we obtain

$$y(x) = 1 + x + 0.5\,x^2 + 0.166667\,x^3 + 0.0416667\,x^4 + 0.00833333\,x^5 + 0.00138889\,x^6 + 0.000198413\,x^7$$
$$+0.0000248016\,x^8 + 2.75573\,10^{-6}\,x^9 + 2.75573\,10^{-7}\,x^{10},$$

and the exact solution $y(x) = e^x$ by Taylor series of order 10 is

$$y(x) = 1 + x + 0.5\,x^2 + 0.166667\,x^3 + 0.0416667\,x^4 + 0.00833333\,x^5 + 0.00138889\,x^6 + 0.000198413\,x^7$$
$$+0.0000248016\,x^8 + 2.75573\,10^{-6}\,x^9 + 2.75573\,10^{-7}\,x^{10}.$$

4. Conclusion

In this article, we offer a new method for solving different kinds of Emden -Fowler equations of higher order with boundary conditions by applying a suggested modification of ADM. The results obtained using the presented method were very accurate compared to some modifications made on the ADM, and very close to the exact solution as we noted in the illustrative examples. Moreover, the exact solution was obtained several times as in the examples (8,9). Figs. 1-6 show that the approximate solution curves match favourably well with the exact solution curves. The MADM was able to solve these type of equations that the standard of the ADM could not solve.The numerical and graphical results depict the efficiency and accuracy of the proposed method.

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Competing Interests

Authors have declared that no competing interests exist.

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