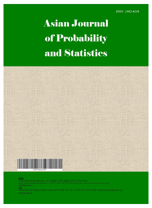

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A Note on Second Order Slope Rotatable Designs under Intra-class Correlated Errors Using Pairwise Balanced Designs

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Authors' contributions

This work was carried out in collaboration among all authors. Author KR designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BS and BRV managed the analyses of the study. Author BRV managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, a study on second order slope rotatable designs under intra-class correlation structure of errors using pairwise balanced designs is suggested. Further, the variance function of the estimated slopes for different values of intra-class correlated coefficient 'ρ', $(-0.9 \le \rho \le 0.9)$ for $6 \le v \le 15$ (v- number of factors) are studied.

Keywords: Response surface designs; slope rotatability; intra-class correlated errors.

1 Introduction

In the context of response surface methodology, Box and Hunter [1] introduced the concept of rotatability assuming errors are uncorrelated and homoscedastic. Das and Narasimham [2] constructed rotatable designs through balanced incomplete block designs (BIBD). Tyagi [3] constructed SORD using pairwise balanced designs (PBD).

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In response surface methodology, good estimation of the derivatives of the response function may be as important or perhaps more important than estimation of mean response. Estimation of difference in responses at two different points in the factor space will often be of great importance. In differences in responses at two points close together is of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in animal etc. (cf. [4]). Hader and Park [5] introduced slope rotatable central composite designs. Park [4] studied a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham [6,7] constructed second order slope rotatable designs (SOSRD) through BIBD and PBD respectively. Victorbabu [8] suggested a review on SOSRD.

So far all the authors studied rotatable and slope rotatable designs assuming errors to be uncorrelated and homoscedastic. However, it is not uncommon to come across the practical situations when the errors are correlated, violating the usual assumptions. Das [9] introduced and studied robust second order rotatable designs (RSORD). Das [10] studied slope rotatability with correlated errors. Rajyalakshmi [11] suggested some contributions to second order rotatable and slope rotatable designs under different correlated error structures. Rajyalakshmi and Victorbabu [12,13,14] constructed SOSRD under intra-class correlation structure of errors using central composite designs, symmetrical unequal block arrangements (SUBA) with two unequal block sizes and BIBD respectively. Sulochana and Victorbabu [15,16,17] studied SOSRD under intra-class correlation structure of errors using central composite designs, symmetrical unequal block arrangements (SUBA) with two unequal block sizes and BIBD respectively.

In this paper following the works of Das [10], Rajyalakshmi and Victorbabu [12,13,14], here a study of SOSRD under intra-class correlated structure of errors using PBD is suggested. Further we study the variance function of the estimated slopes for different values of intra-class correlated coefficient (ρ) for 6 ≤ v ≤ 15 (v number of factors) is obtained.

2 Conditions for SOSRD under Intra-class Correlated Structure of Errors

A second order response surface design $D=\left(\left(X_{iu}\right)\right)$ for fitting,

$$Y_u\left(x\right)=b_0+\sum_{i=1}^{v} b_i X_{iu}+\sum_{i=1}^{v} b_{ii} X_{iu}^2+\sum_{i=1}^{v}\sum_{i<j}^{v} b_{ij} X_{iu} X_{ju}+e_u \tag{2.1}$$

where X_{iu} denotes the level of the ith factor (i=1,2,…,v) in the uth run (u=1,2,…,N) of the experiment, e_u's are correlated random errors, is said to be a SOSRD under intra-class correlated structure of errors, if the variance of the estimate of first order partial derivative of $Y_u\left(X_{1u},X_{2u},X_{3u},...,X_{vu}\right)$ with respect to each independent variable X_i is only a function of the distance $\left(d^2=\sum_{i=1}^{v} X_i^2\right)$ of the point $\left(X_{1u},X_{2u},X_{3u},...,X_{vu}\right)$ from the origin (centre of the design). Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions (cf. Das [9,10], Rajyalakshmi [11], Rajyalakshmi and Victorbabu [12,13,14]).

$$\sum_{u-1}^{N}\prod_{i=1}^{v}X_{iu}^{\alpha_i}=0 \text{ if } \alpha_i \text{is odd, for } \sum \alpha_i \leq 4 \tag{2.2}$$

$$\sum_{u=1}^{N}X_{iu}^{2}=\text{constant}=N\mu_2 \tag{2.3}$$

$$\sum_{u=1}^{N}X_{iu}^{4}=\text{constant}= cN\mu_4, \text{ for all } i \tag{2.4}$$

$$\sum_{u=1}^{N}X_{iu}^{2}X_{ju}^{2}=\text{constant}=N\mu_4, \text{ for all values } i \neq j \tag{2.5}$$

From (2.4) and (2.5), we have,

$$\sum_{u=1}^{N}X_{iu}^{4}= c\sum_{u=1}^{N}X_{iu}^{2}X_{ju}^{2} \tag{2.6}$$

where c, μ_2 and μ_4 are constants. The summation is over the designs points.

Using the above simple symmetric conditions, the variances and covariances of the estimated parameters under the intra-class correlated structure of errors are as follows:

$$V\left(\hat{b}_0\right)=\frac{[\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2]\sigma^2 A}{N\Delta} \tag{2.7}$$

$$V\left(\hat{b}_i\right)=\frac{\sigma^2(1-\rho)}{N\mu_2} \tag{2.8}$$

$$V\left(\hat{b}_{ij}\right)=\frac{\sigma^2(1-\rho)}{N\mu_4} \tag{2.9}$$

$$V\left(\hat{b}_{ii}\right)=\frac{\sigma^2(1-\rho)[\mu_4\left(c+v-2\right)A-(v-1)\rho N\mu_2^2-(v-1)\mu_2^2(1-\rho)]}{\left(c-1\right)N\mu_4\Delta} \tag{2.10}$$

$$Cov\left(\hat{b}_0,\hat{b}_{ii}\right)=\frac{-\sigma^2\mu_2^2(1-\rho)A}{N\Delta} \tag{2.11}$$

$$Cov\left(\hat{b}_{ii},\hat{b}_{ij}\right)=\frac{\sigma^2(1-\rho)[\mu_2^2\left(1-\rho\right)-\mu_4 A+\rho N\mu_2^2]}{\left(c-1\right)N\mu_4\Delta} \tag{2.12}$$

where $A=\left\{1+\left(N-1\right)\rho\right\}$, $\Delta=\left[\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2-v\mu_2^2\left(1-\rho\right)\right]$

and the other covariances are zero.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a non-singular second order slope rotatable design under intra-class correlated structure of errors is

$$\mu_4\left(c+v-1\right)A-v\rho N\mu_2^2-v\mu_2^2\left(1-\rho\right)>0 \tag{2.13}$$

From (2.13), we have,

$$\frac{\mu_4}{\mu_2^2}>\frac{v}{c+v-1} \quad \text{(non-singularity condition)} \tag{2.14}$$

If the non-singularity condition (2.14) exists then only the design exists.

For the second order model,

$$\frac{\partial\hat{Y}}{\partial x_i}=\hat{b}_i+2\hat{b}_{ii}X_i+\sum_{i=1,j\neq i}^{v}\hat{b}_{ij}X_j,$$

$$V\left(\frac{\partial\hat{Y}}{\partial x_i}\right)=V\left(\hat{b}_i\right)+4X_i^2V\left(\hat{b}_{ii}\right)+\sum_{i=1,j\neq i}^{v}X_j^2V\left(\hat{b}_{ij}\right) \tag{2.15}$$

The condition for right hand side of equation (2.15) to be a function of $(d^2=\sum_{i=1}^{v}X_i^2)$ alone (for slope rotatability) is clearly,

$$V\left(\hat{b}_{ii}\right)=\frac{1}{4}V\left(\hat{b}_{ij}\right) \tag{2.16}$$

On simplification of (2.16) using (2.9) and (2.10) leads to

$$\left(\frac{AcN\mu_4-B}{1-\rho}\right)\left[4N-\left(\frac{AcN\mu_4-B}{A\mu_4}\right)v\left(\frac{N\mu_2^2\left(1-\rho\right)}{A\mu_4}\right)-\left(v-2\right)\left(\frac{AN\mu_4-B}{A\mu_4}\right)\right]+$$
$$\left(\frac{AN\mu_4-B}{1-\rho}\right)\left[4\left(v-2\right)-\left(v-1\right)\left(\frac{AN\mu_4-B}{AN\mu_4}\right)\right]-N^2\mu_2^2\left[4\left(v-1\right)+v\left(\frac{AN\mu_4-B}{AN\mu_4}\right)\right]=0 \tag{2.17}$$

$A=\{1+(N-1)\rho\}$, $B=\rho N^2\mu_2^2$

If ρ = 0 condition (2.17) is equal to

$$\mu_4\left[v(5\text{-}c)\text{-}(c\text{-}3)^2\right]+\mu_2^2\left[v(c\text{-}5)+4\right]=0 \tag{2.18}$$

which is similar to the SOSRD condition of Victorbabu and Narasimham (1991a).

Therefore, equations (2.2) to (2.12), (2.14) to (2.17) give a set of conditions for SOSRD under intra-class correlated structure of errors for any general second order response surface design. Further,

$$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)=\frac{1}{N}\left(\frac{1}{\mu_2}+\frac{d^2}{\mu_4}\right)(1\text{-}\rho)\sigma^2 \tag{2.19}$$

3 A Study on SOSRD under Intra-class Correlation Structure of Errors Using PBD

Following the methods of constructions, of Das [9,10], Victorbabu and Narasimham [6,7], Rajyalakshmi and Victorbabu [12,13,14]. Study of SOSRD under intra-class correlated structure of errors using pairwise balanced designs involves ρ be the correlation errors of any two observations, each having the same variance σ^2.

Let $(v, b, r, k_1, k_2, ... k_p, \lambda)$ denote a PBD and $k=\sup(k_1, k_2, ..., k_p)$, $2^{t(k)}$ denote a fractional replicate of 2^k in +1 and -1 levels, in which no interaction with less than five factors is confounded and n_0 be the number of central points in the design.

Theorem: The design points, $\left[1-(v, b, r, k_1, k_2, ... k_p, \lambda)\right]2^{t(k)} \cup (a, 0, ..., 0)2^1 \cup (n_0)$ will give a v-dimensional SOSRD with intra-class correlation structure of errors using PBD in design points $N=b2^{t(k)}+2v+n_0$, where a^2 is positive real root of the fourth degree polynomial equation,

$$\begin{aligned}
&\left[(8v\text{-}4N)\right]A^2a^8+\left[8vr2^{t(k)}\right]A^2a^6+\\
&\left[2vr^2 2^{2t(k)}+\left\{\left((12\text{-}2v)\lambda\text{-}4r\right)N+\left(16\lambda\text{-}20v\lambda+4vr\right)\right\}2^{t(k)}\right]A^2a^4+\\
&\left[4vr^2+(16\text{-}20v)r\lambda\right]A^2 2^{2t(k)}a^2+\left[(5v\text{-}9)\lambda^2+(6\text{-}v)r\lambda\text{-}r^2\right]A^2N2^{2t(k)}+\\
&\left[vr+4\lambda\text{-}5v\lambda\right]A^2r^2 2^{3t(k)}=0
\end{aligned} \tag{3.1}$$

If at least one positive real root exists for equation (3.1) then only the design exists.

Proof: For the design points, $[1\text{-}(v,b,r,k_1, k_2,...k_p,\lambda)]2^{t(k)}\ \text{U}\ (a,0,0,....0)2^1\ \text{U}\ (n_0)$ generated from PBD in $N=b2^{t(k)}+2vn_a+n_0$ design points, simple symmetry conditions (2.2) to (2.5) are true. From (2.3) to (2.5), we have,

$$\sum_{u=1}^{N} X_{iu}^2 = r2^{t(k)} + 2a^2 = N\mu_2 \tag{3.2}$$

$$\sum_{u=1}^{N} X_{iu}^4 = r2^{t(k)} + 2a^4 = cN\mu_4 \tag{3.3}$$

$$\sum_{u=1}^{N} X_{iu}^2 X_{ju}^2 = \lambda 2^{t(k)} = N\mu_4 \tag{3.4}$$

By substituting (3.2), (3.3) and (3.4) in (2.17) and on simplification, we get (3.1).The design exists only if at least one positive real root exists for equation (3.1). Solving (3.1) we get SOSRD with intra-class correlated structure of errors using PBD with different values of 'a' for the 'v' different factors. The variance of estimated slopes of these SOSRD under intra-class correlated structure of errors for $0 \leq \rho \leq 0.9$ and for $6 \leq v \leq 15$ factors are given in Table 1.

Example 1: We illustrate the construction of SOSRD under intra-class correlated structure of errors for v=6 factors with the help of PBD with parameters $(v=6, b=7, r=3, k_1=3,\ k_2=2, \lambda=1)$.

The design points, $[1\text{-}(v=6,b=7,r=3,k_1=3,\ k_2=2,\lambda=1)]2^{t(k)}U(a,0,0,....0)2^1\ U\ (n_0=1)$ will give a SOSRD under intra-class structure of errors using a PBD in N=69 design points. We have,

$$\sum_{u=1}^{N} X_{iu}^2 = 24 + 2a^2 = N\mu_2 \tag{3.5}$$

$$\sum_{u=1}^{N} X_{iu}^4 = 24 + 2a^4 = cN\mu_4 \tag{3.6}$$

$$\sum_{u=1}^{N} X_{iu}^2 X_{ju}^2 = 8 = N\mu_4 \tag{3.7}$$

From (3.5), (5.6) and (5.7), we get $\mu_2 = \frac{24+2a^2}{69},\ \mu_4 = \frac{8}{69}$ and $c = \frac{24+2a^4}{8}$. Substituting μ_2, μ_4 and c in (2.17) and on simplification, we get the following fourth degree polynomial equation in a^2.

$$\left(1+68\rho\right)^2 \left(228a^8 - 1152a^6 - 32a^4 + 6144a^2 - 16128\right) = 0 \tag{3.8}$$

Equation (3.8) has only one positive real root a^2 = 4.5314 $((\forall(\frac{1}{N-1} < \rho < 1)))$. This can be alternatively written directly from (3.1). By substituting this 'a' value in (3.5), (3.6) and (3.7) on simplification we get $\mu_2 = 0.4792,\ \mu_4 = 0.1159$, c = 8.1333. From (2.14) non-singularity condition (0.5047 > 0.4569) is also satisfied. From (2.7) to (2.12), we obtain the variances and covariances. Further, from (2.19), we get,

$$V\left(\frac{\partial \hat{y}}{\partial x_i}\right)=\left(0.0302+0.125d^2\right)(1\text{-}\rho)\sigma^2. \tag{3.9}$$

Example 2: The design points, $[1\text{-}(v=15, b=16, r=6, k_1=6, k_2=5, \lambda=2)]2^{t(k)}\ U\,(a,0,0,....0)2^1\ U\,(n_0=1)$ will give a SOSRD with intra-class structure of errors using a PBD in N=543 design points for six factors. We have

$$\sum_{u=1}^{N} X_{iu}^2=192+2a^2=N\mu_2 \tag{3.10}$$

$$\sum_{u=1}^{N} X_{iu}^4=192+2a^4=cN\mu_4 \tag{3.11}$$

$$\sum_{u=1}^{N} X_{iu}^2 X_{ju}^2=64=N\mu_4 \tag{3.12}$$

From (3.10), (3.11) and (3.12), we get $\mu_2=\frac{192+2a^2}{543},\ \mu_4=\frac{64}{543}$ and $c=\frac{192+2a^4}{64}.$ Substituting μ_2, μ_4 and c in (2.17) and on simplification, we get the following biquadratic equation in a^2.

$$\left(1+542\rho\right)^2\left(2052a^8\text{-}23040a^6\text{-}56704a^4+1277952a^2-5382144\right)=0 \tag{3.13}$$

Equation (3.8) has only one positive real root a^2 = 10.4717 $\left(\left(\forall\left(\frac{1}{N\text{-}1}<\rho<1\right)\right)\right)$. This can be alternatively written directly from (3.1). By substituting this 'a' value in (3.5), (3.6) and (3.7) on simplification we get $\mu_2=0.3922,\ \mu_4=0.1159$, c = 6.4268. From (2.14) non-singularity condition (0.7536 >0.7343) is also satisfied. From (2.7) to (2.12), we obtain the variances and covariances. Further, from (2.19), we get,

$$V\left(\frac{\partial \hat{y}}{\partial x_i}\right)=\left(0.0047+0.0156d^2\right)(1\text{-}\rho)\sigma^2. \tag{3.14}$$

Note: We may point out here that this SOSRD with intra-class correlated structure of errors using PBD with parameters (v=6,b=7,r=3,k_1=3,k_2=2,λ=1) has only 69 design points for 6 factors, whereas the corresponding SOSRD with intra-class correlated structure of errors using BIBD with parameters (v=6, b=15, r=5, k=2,λ=1) obtained by Rajyalakshmi and Victorbabu [14] needs 73 design points and Rajyalakshmi and Victorbabu [13] symmetrical unequal block arrangements with two unequal block sizes with parameters (v=6, b=7, r=3, k_1=2, k_2=3, b_1=3, b_2=4, λ=1) by Rajyalakshmi and Victorbabu [11] needs 69 design points. Thus the method leads to a 6-factor SOSRD with intra-class structure of errors in less number of design points than the corresponding SOSRD with intra-class structure of errors using BIBD.

Table 1. The variances of estimated derivatives (slopes) for $6 \le v \le 15$ factors of SOSRD under intra-class correlated structure of errors using PBD

ρ	**(v=6, b=7, r=3, k_1=3, k_2=3, λ=1) N = 69, a = 2.1287**	**(v=8, b=15, r=6, k_1=4, k_2=3, k_3=2, λ=2) N =257 , a = 2.7066**	**(v=9, b=11, r=5, k_1=5, k_2=4, k_3=3, λ=2) N = 195, a = 2.8386**
	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$
0	$0.0302\sigma^2+0.125\ \sigma^2d^2$	$0.009\ \sigma^2+0.0313\ \sigma^2d^2$	$0.0104\ \sigma^2+0.0313\ \sigma^2d^2$
0.1	$0.0272\ \sigma^2+0.1125\ \sigma^2d^2$	$0.0081\ \sigma^2+0.0281\ \sigma^2d^2$	$0.0094\sigma^2+0.0281\ \sigma^2d^2$
0.2	$0.0242\ \sigma^2+0.1\ \sigma^2d^2$	$0.0072\ \sigma^2+0.025\ \sigma^2d^2$	$0.0083\ \sigma^2+0.025\ \sigma^2d^2$
0.3	$0.0212\ \sigma^2+0.0875\ \sigma^2d^2$	$0.0063\ \sigma^2+0.0219\ \sigma^2d^2$	$0.0073\ \sigma^2+0.0219\ \sigma^2d^2$
0.4	$0.0181\ \sigma^2+0.075\ \sigma^2d^2$	$0.0054\ \sigma^2+0.0188\ \sigma^2d^2$	$0.0062\ \sigma^2+0.0188\sigma^2d^2$
0.5	$0.0151\ \sigma^2+0.0625\ \sigma^2d^2$	$0.0045\ \sigma^2+0.0156\ \sigma^2d^2$	$0.0052\ \sigma^2+0.0156\ \sigma^2d^2$
0.6	$0.0121\ \sigma^2+0.05\ \sigma^2d^2$	$0.0036\ \sigma^2+0.0125\ \sigma^2d^2$	$0.0042\ \sigma^2+0.0125\sigma^2d^2$
0.7	$0.0091\ \sigma^2+0.0375\ \sigma^2d^2$	$0.0027\ \sigma^2+0.0094\ \sigma^2d^2$	$0.0031\ \sigma^2+0.0095\ \sigma^2d^2$
0.8	$0.016\ \sigma^2+0.0025\ \sigma^2d^2$	$0.0018\ \sigma^2+0.0063\ \sigma^2d^2$	$0.0021\sigma^2+0.0063\ \sigma^2d^2$
0.9	$0.003\ \sigma^2+0.0125\ \sigma^2d^2$	$0.0009\ \sigma^2+0.0031\ \sigma^2d^2$	$0.001\sigma^2+0.0031\ \sigma^2d^2$

ρ	**(v=10, b=11, r=5, k_1=5, k_2=4, λ=2) N =197 , a = 2.8928**	**(v=12, b=16, r=6, k_1=6, k_2=5, k_3=4, k_4=3, λ=2) N = 537, a = 3.1055**	**(v=13, b=16, r=6, k_1=6, k_2=5, k_3=4, k_4=3, λ=2) N = 539, a = 3.1416**
	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$
0	$0.0103\ \sigma^2+0.0313\ \sigma^2d^2$	$0.0047\ \sigma^2+0.0156\ \sigma^2d^2$	$0.0047\ \sigma^2+0.0156\ \sigma^2d^2$
0.1	$0.0093\ \sigma^2+0.0282\ \sigma^2d^2$	$0.0043\ \sigma^2+0.0141\ \sigma^2d^2$	$0.0043\ \sigma^2+0.0141\ \sigma^2d^2$
0.2	$0.0083\ \sigma^2+0.025\ \sigma^2d^2$	$0.0038\sigma^2+0.0125\ \sigma^2d^2$	$0.0038\sigma^2+0.0125\ \sigma^2d^2$
0.3	$0.0072\ \sigma^2+0.0219\ \sigma^2d^2$	$0.0033\ \sigma^2+0.0109\ \sigma^2d^2$	$0.0033\ \sigma^2+0.0109\ \sigma^2d^2$
0.4	$0.0062\ \sigma^2+0.0188\ \sigma^2d^2$	$0.0028\ \sigma^2+0.0094\ \sigma^2d^2$	$0.0028\ \sigma^2+0.0094\ \sigma^2d^2$
0.5	$0.0052\ \sigma^2+0.0157\ \sigma^2d^2$	$0.0024\ \sigma^2+0.0078\ \sigma^2d^2$	$0.0024\ \sigma^2+0.0078\ \sigma^2d^2$
0.6	$0.0041\ \sigma^2+0.0125\ \sigma^2d^2$	$0.0019\ \sigma^2+0.0063\ \sigma^2d^2$	$0.0019\ \sigma^2+0.0063\ \sigma^2d^2$
0.7	$0.0031\sigma^2+0.0094\ \sigma^2d^2$	$0.0014\ \sigma^2+0.0047\ \sigma^2d^2$	$0.0014\ \sigma^2+0.0047\ \sigma^2d^2$
0.8	$0.0021\ \sigma^2+0.0063\ \sigma^2d^2$	$0.0009\sigma^2+0.0031\ \sigma^2d^2$	$0.0009\sigma^2+0.0031\ \sigma^2d^2$
0.9	$0.001\ \sigma^2+0.0031\ \sigma^2d^2$	$0.0005\sigma^2+0.0016\ \sigma^2d^2$	$0.0005\sigma^2+0.0016\ \sigma^2d^2$

ρ	**(v=14, b=16, r=6, k_1=6,k_2=5,k_3=4,λ=2) N = 541, a = 3.1847**	**(v=15, b=16, r=6, k_1=6,k_2=5,λ=2) N = 543, a = 3.2360**
	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial X_i}\right)$
0	$0.0047\ \sigma^2+0.0156\ \sigma^2d^2$	$0.0047\ \sigma^2+0.0156\ \sigma^2d^2$
0.1	$0.0043\ \sigma^2+0.0141\ \sigma^2d^2$	$0.0043\ \sigma^2+0.0141\ \sigma^2d^2$
0.2	$0.0038\sigma^2+0.0125\ \sigma^2d^2$	$0.0038\sigma^2+0.0125\ \sigma^2d^2$
0.3	$0.0033\ \sigma^2+0.0109\ \sigma^2d^2$	$0.0033\ \sigma^2+0.0109\ \sigma^2d^2$
0.4	$0.0028\ \sigma^2+0.0094\ \sigma^2d^2$	$0.0028\ \sigma^2+0.0094\ \sigma^2d^2$
0.5	$0.0024\ \sigma^2+0.0078\ \sigma^2d^2$	$0.0024\ \sigma^2+0.0078\ \sigma^2d^2$
0.6	$0.0019\ \sigma^2+0.0063\ \sigma^2d^2$	$0.0019\ \sigma^2+0.0063\ \sigma^2d^2$
0.7	$0.0014\ \sigma^2+0.0047\ \sigma^2d^2$	$0.0014\ \sigma^2+0.0047\ \sigma^2d^2$
0.8	$0.0009\sigma^2+0.0031\ \sigma^2d^2$	$0.0009\sigma^2+0.0031\ \sigma^2d^2$
0.9	$0.0005\sigma^2+0.0016\ \sigma^2d^2$	$0.0005\sigma^2+0.0016\ \sigma^2d^2$

Table 2. Study of dependence of estimated slope of second order response surface design under intra-class correlated structure of errors PBD at different design points for $6 \le v \le 15$ factors for different values of 'ρ', 'd' and σ=1

(v=6, b=7, r=3, k_1=3, k_2=3, λ=1)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0315	0.0352	0.0415	0.0502	0.0615	0.0752	0.0915	0.1102	0.1315	0.1552
0.1	0.0283	0.0317	0.0373	0.0452	0.0553	0.0677	0.0823	0.0992	0.1183	0.1397
0.2	0.0252	0.0282	0.0332	0.0402	0.0492	0.0602	0.0732	0.0882	0.1052	0.1242
0.3	0.0220	0.0247	0.0291	0.0352	0.0431	0.0527	0.0641	0.0772	0.0921	0.1087
0.4	0.0188	0.0211	0.0249	0.0301	0.0369	0.0551	0.0549	0.0661	0.0789	0.0931
0.5	0.0157	0.0176	0.0207	0.0251	0.0307	0.0376	0.0457	0.0551	0.0657	0.0776
0.6	0.0126	0.0141	0.0166	0.0201	0.0246	0.0301	0.0366	0.0441	0.0526	0.0621
0.7	0.0095	0.0106	0.0125	0.0151	0.0185	0.0226	0.0275	0.0331	0.0395	0.0466
0.8	0.0063	0.007	0.008	0.01	0.0123	0.015	0.0183	0.022	0.0263	0.031
0.9	0.0031	0.0035	0.0041	0.0005	0.0061	0.0075	0.0091	0.011	0.0131	0.0155

(v=8, b=15, r=6, k_1=4, k_2=3, k_3=2, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0093	0.0103	0.0119	0.014	0.0168	0.0203	0.0243	0.029	0.0343	0.0403
0.1	0.0084	0.0092	0.0106	0.0126	0.0151	0.0182	0.0219	0.0261	0.0309	0.0362
0.2	0.0075	0.0082	0.0095	0.0112	0.0135	0.0162	0.0195	0.0232	0.0275	0.0322
0.3	0.0065	0.0072	0.0083	0.0098	0.0118	0.0142	0.017	0.0203	0.0240	0.0282
0.4	0.0056	0.0062	0.0071	0.0084	0.0101	0.0121	0.0146	0.0174	0.0206	0.0242
0.5	0.0047	0.0051	0.0059	0.0069	0.0084	0.0101	0.0121	0.0145	0.0171	0.0201
0.6	0.0037	0.0041	0.0047	0.0056	0.0067	0.0081	0.0097	0.0116	0.0137	0.0161
0.7	0.0028	0.0031	0.0035	0.0042	0.0051	0.0061	0.0073	0.0087	0.0103	0.0121
0.8	0.0019	0.0021	0.0024	0.0028	0.0034	0.0041	0.0049	0.0058	0.0069	0.0081
0.9	0.0009	0.0010	0.0012	0.0014	0.0017	0.002	0.0024	0.0029	0.0034	0.004

(v=9, b=11, r=5, k_1=5, k_2=4, k_3=3, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0107	0.017	0.0132	0.0154	0.0182	0.0217	0.0257	0.0304	0.0358	0.0417
0.1	0.0097	0.0105	0.0119	0.0139	0.0164	0.0195	0.0232	0.0274	0.0322	0.0375
0.2	0.0086	0.0093	0.0106	0.0123	0.0146	0.0173	0.0206	0.024	0.0286	0.0333
0.3	0.0075	0.0082	0.0093	0.0108	0.0128	0.0152	0.018	0.0213	0.025	0.0292
0.4	0.0064	0.0069	0.0079	0.0092	0.0109	0.0129	0.0154	0.0182	0.0214	0.025
0.5	0.0054	0.0058	0.0066	0.0077	0.0091	0.0108	0.0128	0.0152	0.0178	0.0208
0.6	0.0043	0.0047	0.0053	0.0062	0.0073	0.0087	0.0103	0.0122	0.0143	0.0167
0.7	0.0032	0.0035	0.0039	0.0046	0.005	0.0065	0.0078	0.0092	0.0108	0.0126
0.8	0.0021	0.0024	0.0027	0.0031	0.0037	0.0044	0.0052	0.0061	0.0072	0.0084
0.9	0.001	0.0011	0.0013	0.0015	0.0018	0.0021	0.0025	0.0029	0.0035	0.0041

(v=10, b=11, r=5, k_1=5, k_2=4, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0106	0.0116	0.0131	0.0153	0.0181	0.0216	0.0256	0.0303	0.0357	0.0416
0.1	0.0096	0.0104	0.0118	0.0138	0.0164	0.0195	0.0231	0.0273	0.0321	0.0375
0.2	0.0086	0.0093	0.0106	0.0123	0.0146	0.0173	0.0206	0.0243	0.0286	0.0333
0.3	0.0074	0.0081	0.0092	0.0107	0.0127	0.0151	0.0179	0.0212	0.0249	0.0291
0.4	0.0064	0.0069	0.0079	0.0092	0.0109	0.0129	0.0154	0.0182	0.0214	0.025
0.5	0.0054	0.0058	0.0066	0.0077	0.0091	0.0109	0.0129	0.0152	0.0179	0.0209
0.6	0.0042	0.0046	0.0052	0.0061	0.0072	0.0086	0.0102	0.0121	0.0142	0.0166
0.7	0.0032	0.0035	0.0039	0.0046	0.0055	0.0065	0.0077	0.0091	0.0107	0.0125
0.8	0.0022	0.0024	0.0027	0.0031	0.0037	0.0044	0.0052	0.0061	0.0072	0.0084
0.9	0.0010	0.0011	0.0013	0.0015	0.0018	0.0021	0.0025	0.0029	0.0035	0.0041

(v=12, b=16, r=6, k_1=6, k_2=5, k_3=4, k_4=3, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0049	0.0053	0.0061	0.0072	0.0086	0.0103	0.0123	0.0147	0.0173	0.0203
0.1	0.0044	0.0049	0.0056	0.0066	0.0078	0.0093	0.0112	0.0113	0.0157	0.0184
0.2	0.0039	0.0043	0.0049	0.0058	0.0069	0.0083	0.0099	0.0118	0.0139	0.0163
0.3	0.0034	0.0037	0.0043	0.005	0.006	0.0072	0.0086	0.0103	0.0121	0.0142
0.4	0.0029	0.0032	0.0036	0.0043	0.0052	0.0062	0.0074	0.0088	0.0104	0.0122
0.5	0.0025	0.0027	0.0031	0.0036	0.0044	0.0052	0.0062	0.0074	0.0087	0.0102
0.6	0.0019	0.0022	0.0025	0.0029	0.0035	0.0042	0.0049	0.0059	0.007	0.0082
0.7	0.0014	0.0016	0.0018	0.0022	0.0026	0.0031	0.0037	0.0044	0.0052	0.0061
0.8	0.0009	0.001	0.0012	0.0014	0.0017	0.002	0.0024	0.0029	0.0034	0.004
0.9	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	0.0013	0.0015	0.0018	0.0021

(v=13, b=16, r=6, k_1=6, k_2=5, k_3=4, k_4=3, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0049	0.0053	0.0061	0.0072	0.0086	0.0103	0.0123	0.0147	0.0173	0.0203
0.1	0.0044	0.0049	0.0056	0.0066	0.0078	0.0093	0.0112	0.0113	0.0157	0.0184
0.2	0.0039	0.0043	0.0049	0.0058	0.0069	0.0083	0.0099	0.0118	0.0139	0.0163
0.3	0.0034	0.0037	0.0043	0.005	0.006	0.0072	0.0086	0.0103	0.0121	0.0142
0.4	0.0029	0.0032	0.0036	0.0043	0.0052	0.0062	0.0074	0.0088	0.0104	0.0122
0.5	0.0025	0.0027	0.0031	0.0036	0.0044	0.0052	0.0062	0.0074	0.0087	0.0102
0.6	0.0019	0.0022	0.0025	0.0029	0.0035	0.0042	0.0049	0.0059	0.007	0.0082
0.7	0.0014	0.0016	0.0018	0.0022	0.0026	0.0031	0.0037	0.0044	0.0052	0.0061
0.8	0.0009	0.001	0.0012	0.0014	0.0017	0.002	0.0024	0.0029	0.0034	0.004
0.9	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	0.0013	0.0015	0.0018	0.0021

(v=14, b=16, r=6, k_1=6, k_2=5, k_3=4, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0049	0.0053	0.0061	0.0072	0.0086	0.0103	0.0123	0.0147	0.0173	0.0203
0.1	0.0044	0.0049	0.0056	0.0066	0.0078	0.0093	0.0112	0.0113	0.0157	0.0184
0.2	0.0039	0.0043	0.0049	0.0058	0.0069	0.0083	0.0099	0.0118	0.0139	0.0163
0.3	0.0034	0.0037	0.0043	0.005	0.006	0.0072	0.0086	0.0103	0.0121	0.0142
0.4	0.0029	0.0032	0.0036	0.0043	0.0052	0.0062	0.0074	0.0088	0.0104	0.0122
0.5	0.0025	0.0027	0.0031	0.0036	0.0044	0.0052	0.0062	0.0074	0.0087	0.0102
0.6	0.0019	0.0022	0.0025	0.0029	0.0035	0.0042	0.0049	0.0059	0.007	0.0082
0.7	0.0014	0.0016	0.0018	0.0022	0.0026	0.0031	0.0037	0.0044	0.0052	0.0061
0.8	0.0009	0.001	0.0012	0.0014	0.0017	0.002	0.0024	0.0029	0.0034	0.004
0.9	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	0.0013	0.0015	0.0018	0.0021

(v=15, b=16, r=6, k_1=6, k_2=5, λ=2)

ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0049	0.0053	0.0061	0.0072	0.0086	0.0103	0.0123	0.0147	0.0173	0.0203
0.1	0.0044	0.0049	0.0056	0.0066	0.0078	0.0093	0.0112	0.0113	0.0157	0.0184
0.2	0.0039	0.0043	0.0049	0.0058	0.0069	0.0083	0.0099	0.0118	0.0139	0.0163
0.3	0.0034	0.0037	0.0043	0.005	0.006	0.0072	0.0086	0.0103	0.0121	0.0142
0.4	0.0029	0.0032	0.0036	0.0043	0.0052	0.0062	0.0074	0.0088	0.0104	0.0122
0.5	0.0025	0.0027	0.0031	0.0036	0.0044	0.0052	0.0062	0.0074	0.0087	0.0102
0.6	0.0019	0.0022	0.0025	0.0029	0.0035	0.0042	0.0049	0.0059	0.007	0.0082
0.7	0.0014	0.0016	0.0018	0.0022	0.0026	0.0031	0.0037	0.0044	0.0052	0.0061
0.8	0.0009	0.001	0.0012	0.0014	0.0017	0.002	0.0024	0.0029	0.0034	0.004
0.9	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	0.0013	0.0015	0.0018	0.0021

4 Conclusions

From Table 1 and 2 we observe that

1. When the values of 'ρ' increases slope rotatability value of 'a' is decreases for all values of the v factors.
2. At $\rho=0$ the variance of estimated derivatives (slopes) of SOSRD under intra-class correlated structure of errors is equal to the SOSRD uncorrelated errors case.
3. For given v and ρ, $V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right)$ increases as d increases.
4. For given v and d, $V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right)$ decreases as ρ increases.
5. In this method, we obtain designs with fewer number of design points in some cases. The implications of fewer number of design points leads to effective and reduced cost of experimentation.

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Competing Interests

Authors have declared that no competing interests exist.

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